INTRODUCTION TO
DIFFERENTIAL EQUATIONS

Dr. Kalipada Maity

Associate Professor
Department of Mathematics
Mugberia Gangadhar Mahavidyalaya, Bhupatinagar
Purba Medinipur-721425, West Bengal, India

To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, heat transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / NBHM/TIFR/SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 21 chapters altogether. In the chapter 1, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter **??** are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter **??**, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter **??**. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter 2, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter 3 is presented the Total (Pfaffian) Differential Equations. In chapter **??**, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter **??**, Inverse Laplace transformation in chapter **??**. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter **??**, Legendre function and Rodrigue formula in Chapter **??**, Chebyshev functions in chapter **??**, Bessel functions in chapter **??** and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Chapter **??** is concerned with Green's function. In chapter **??**, the application of differential equations are developed in a progressive manner. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

Moreover it gives the author great pleasure to inform the reader that the **second edition** of the book has been improved, well -organized, enlarged and made up-to-date as per latest UGC - CBSC syllabus. The following significant changes have been made in the second edition:

- Almost all the chapters have been rewritten in such a way that the reader will not find any difficulty in understanding the subject matter.
- Errors , omissions and logical mistakes of the previous edition have been corrected.
- The exercises of all chapters of the previous edition have been improved, enlarged and well-organized.
- Two new chapters like Green's Functions and Application of Differential Equations have been added in the present edition.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various university, IIT-JAM, GATE, and CSIR-UGC(NET) have been provided in almost every chapters which strongly help the students for their preparation of those examinations from graduate label.

In view of the above mentioned features it is expected that this new edition will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations

is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation, Green's function and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 650 worked out examples
- Large number of exercises
- More than 700 multiple choice questions with answers
- Suitable for New UGC-CBSC syllabus of ODE & PDE
- Suitable for GATE, NET, NBHM, TIFR, JAM, JEST, IAS, SSC examinations.

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I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

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Kalipada Maity,
E-mail: kalipada_maity@yahoo.co.in

Contents

Chapter 1

Fundamental Concept of Differential Equations

1.1 Introduction

Differential equations have wide applications in various science and engineering disciplines. In general, modelling variations of a physical quantity, such as displacement, velocity, pressure,temperature, stress, strain, or concentration of a pollutant with the change of time t or location, such as the coordinates (x, y, z) or both would require differential equations. Similarly, studying the variation of a physical quantity on other physical quantities would lead to differential equations. For example, the change of strain on stress for some viscoelastic materials follows a differential equation. These differential equations may be ordinary or partial. In this chapter, the most basic concepts of differential equations, formation of differential equations, also the order and degree of differential equations are given in details.

1.2 Historical Note

Differential equations first came into existence with the invention of calculus by Isaac Newton(16 42-1727) and Gottfried Wilhelm Leibniz(1646-1716). Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date from 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism, and did not begin to publish his results until 1687 with the appearance of his most famous book, *Philosophiae Naturalis Principia Mathematica*. In the eighteenth century, Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications, while most notably by Euler. In fact, Newton classified first order differential equations according to the forms $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = f(y)$ and $\frac{dy}{dx} = f(x, y)$ and for the latter equation he developed a method of solution using infinite series when $f(x, y)$ is a polynomial in x and y. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was fully conscious of the power of good mathematical notation and he developed the natation , derivative $\frac{dy}{dx}$, the integral sign, introduced the methods of separation of variables in 1691, the reduction of homogeneous equations to separable ones in 1691 and the procedure for solving first order linear equations in 1694. The Jacob Bernoulli(1654- 1705) solved the Bernoulli differential equation in 1695. This is an ordinary differential equation of the form $y' + P(x)y = Q(x)y^n$ for which he obtained exact solutions. In particular, Jacob solve the differential equation $\frac{dy}{dx} = \left[\frac{a^3}{(b^2y-a^3)}\right]^{\frac{1}{2}}$ and in 1694 Johann Bernoulli was able to solve the differential equation $\frac{dy}{dx} = \frac{y}{ax}$ though it was not yet known that $d(\log x) = \frac{dx}{x}$. A problem of importance popularly known as the brachistochrone problem regarding the curve of fastest descent drew the attention of both the brothers. Daniel Bernoulli (1700-1782), son of Johann was a professor of botany and later physics but because of his keep interests in mathematics made substantial contribution in partial differential equations and their applications. The famous Bernoulli equation in fluid dynamics is due to Daniel Bernoulli. He was also the first to encounter the functions that a century later became known as Bessel functions. Leonard Euler(1707-1783), the greatest and the most prolific mathematician of the eighteenth century, was a student of Johann Bernoulli and a friend of Daniel Bernoulli in St. Petersburg Academy. Among others, Euler identified the condition for exactness of first order differential equations, developed the theory of integrating factors and the general solution of homogeneous linear equations with constant coefficients in 1743. He extended these results to non-homogenous equations in 1751. Euler is the first to use power series frequently in solving differential equations and discovered numerical procedure for solving differential equations. The greatest mathematician of the eighteenth century, Leonhard Euler $(1707 - 1783)$, grew up near Basel and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. Euler was the most prolific mathematician of all time, his collected works fill more than 70 large volumes. His interests ranged over all areas of mathematics and many fields of application. Of particular interest here is his formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Eulers work in mechanics, ″The first great work in which analysis is applied to the science of movement″. Among other things, Euler identified the condition for exactness of first order differential equations in $1734 - 35$, developed the theory of integrating factors in the same paper and gave the general solution of homogeneous linear equations with constant coefficients in 1743. He extended the latter results to nonhomogeneous equations in $1750 - 51$. Beginning about 1750, Euler made frequent use of power series in solving differential equations. He also proposed a numerical procedure in $1768 - 69$, made important contributions in partial differential equations and gave the first systematic treatment of the calculus of variations. Joseph Lagrange $(1736 - 1813)$ who succeeded Euler in the chair of Mathematics showed that the general solution of an n^{th} order linear homogenous differential equation is a linear combination of n independent solutions. He solved this problem in 1755 and sent the solution to Euler. Both further developed Lagrange's method and applied it to mechanics, which led to the formulation of Lagrangian mechanics. In 1774 he gave a complete development of the method of variation of parameters. Pierre Simon de Laplace $(1749 - 1827)$ famous for his monumental work on celestial mechanics studied extensively the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, also known as Laplace's equation, in connection with gravitational attraction. Adrien-Marie Legendre $(1752 - 1833)$ held various positions in the French Academie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendres equation, first appeared in 1784 in his study of the attraction of spheroids. Indeed, by the nineteenth century almost all of the standard methods of solving ordinary and partial differential equations were known. The twentieth century discoveries were more towards a systematic and rigorous general theory and numerical solution. Many made significant contributions in this direction and among them Picard, Runge, Kutta, Hilbert and Frobenious special mention.

1.3 Differential Equations

Definition 1.1 An equation involving derivatives or differentials of one or more dependent variables *w.r.t.* one or more independent variables is called a differential Equation.

Some examples of differential equations are

$$\frac{dy}{dx} = \frac{a^3}{(b^2y - a^3)} \tag{1.1}$$

$$\frac{dy}{dx} + 6y = 0 \tag{1.2}$$

$$dy = (x^2 + \sin x)dx \tag{1.3}$$

$$\frac{dy}{dx} + x^2y = 6x^3 \tag{1.4}$$

$$(x+y)^3\frac{dy}{dx} = c \tag{1.5}$$

$$\frac{d^2y}{dx^2} + 5y = 0 \tag{1.6}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + y = 7x^2 \tag{1.7}$$

$$\frac{d^2y}{dx^2} = 6\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \tag{1.8}$$

$$x^3\frac{d^2y}{dx^2} + 3\cos x\frac{dy}{dx} + 5\sin xy = 0 \tag{1.9}$$

$$4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + x^3\sin y = 0 \tag{1.10}$$

$$\frac{\partial^2u}{\partial t^2} = c^2\frac{\partial^2u}{\partial x^2} \tag{1.11}$$

$$\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = 0 \tag{1.12}$$

$$\frac{\partial^2u}{\partial x^2} = 4\frac{\partial u}{\partial t} \tag{1.13}$$

$$\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} + \frac{\partial^2u}{\partial z^2} = 0 \tag{1.14}$$

$$\frac{\partial^2u}{\partial t^2} = k\left(\frac{\partial^3u}{\partial x^3}\right)^2 \tag{1.15}$$

Hypothesis : Let $\mathfrak{I} \subseteq \mathfrak{R}^{n+1}$ be a domain and $\mathbb{I} \subseteq \mathfrak{R}$ be an interval. Let $F : \mathbb{I} \times \mathfrak{I} \rightarrow \mathfrak{R}$ be a function defined by $(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}) \mapsto F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n})$ such that F is not a constant function in the variable $\frac{d^ny}{dx^n}$.

Definition 1.2 (**Ordinary Differential Equations(ODE)**) Let the above hypothesis on F be satisfied. An differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation. Let x be an independent variable and y be a dependent variable then most general ordinary differential equation can be written as

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0 \tag{1.16}$$

Equations (1.1) to (1.10) all are examples of ordinary differential equations.

Definition 1.3 (**Partial Differential Equations (PDE)**) A partial differential equation (PDE) is an equation involving derivatives of an unknown function $u : \Omega \rightarrow \Re$, where Ω is an open subset of $\Re^d$, $d \geq 2$ (or, more generally, of a differentiable manifold of dimension $d \geq 2$).
In case $d = n$, the dependent variable u is a function of more than one independent variable, say $x_1, x_2, \cdots, x_n$, then the partial differential equation for the function u is an equation of the form

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \cdots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \cdots\right) = 0.$$

Equations (1.11) to (1.15) are examples of partial differential equations.

1.4 Order and Degree

Definition 1.4 (Order:) The **order** of a differential equation is the same as the order of the highest derivative used in that differential equation. Differential equations (1.1)-(1.5) are of first order. Differential equations (1.6) to (1.14) are of second order. Differential equation (1.15) is of third order.

Definition 1.5 (Degree:) The **degree** of a differential equation is the greatest exponent of the highest ordered derivative involving in it, when the equation is freed from radicals and fractional powers with respect to the derivatives. To study the degree of a differential equation, the key point is that the differential equation must be a polynomial in derivatives i.e., $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc. Therefore when an equation is a polynomial in all the derivatives(or differential coefficients) involved, the power to which the highest derivative(differential coefficient) is raised as known as the degree of the equation. Also it may be mention here that the order and degree (if defined) of a differential equation are always positive integers. Differential equations (1.1) to (1.6) and (1.9) to (1.14) are of first degree. Differential equations (1.7), (1.8) and (1.15) are of second degree.

Remark: It may be again mentioned here that the degree of most ordinary differential equation may not be defined. Sometimes it is observed that when an ordinary differential equation is reduced to expressed in integral powers of the derivatives, then the resulting equation is changed and its solutions will also be changed. So, after this reduction, the degree of the original differential equation can not be obtained.

For example the degree and order of the differential equation $e^{\frac{d^2y}{dx^2}} = x + 3$ are one and two respectively, as it can be expressed as $\frac{d^2y}{dx^2} = \log_e(x+3)$. But the degree of the differential equation $\frac{d^2y}{dx^2} + \sin(\frac{d^2y}{dx^2}) + y = 0$ can not be defined as it is not a polynomial of derivatives, although it has order 2. Similarly, the degree of $\frac{d^3y}{dx^3} + y^2 + e^{\frac{dy}{dx}} = 0$ is not defined as the differential equation is not a polynomial equation in its derivatives although it has order 3.

Example 1.1 Determine the order and degree of the following ODEs.

(i)$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \rho\frac{d^2y}{dx^2}$ **N.B.U(Hons)-08** (ii)$\left(\frac{d^2y}{dx^2}\right)^2 + y = \frac{dy}{dx}$ **N.B.U(Hons)-07**

(iii)$(x+y)^2\frac{dy}{dx} + 5y = 3x^4$ (iv)$\frac{dy}{dx} + \sin(\frac{dy}{dx}) = 0$ (v)$\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx} + \sin y = 0$

(vi)$\left\{\frac{d^3y}{dx^3}\right\}^{\frac{3}{2}} + \left\{\frac{d^3y}{dx^3}\right\}^{\frac{2}{3}} = 0$ (vii)$\left(\frac{d^2y}{dx^2}\right)^{-\frac{7}{2}}\frac{dy}{dx} + y\left(\frac{d^2y}{dx^2}\right)^{-\frac{5}{2}} = 0$

Solution. (i) Here, $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \rho\frac{d^2y}{dx^2}$ or $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = \rho^2\left(\frac{d^2y}{dx^2}\right)^2$.
So the order and degree of the equation are two, since the highest order derivative is two and the exponent of the highest order derivative is also two.
(ii) The order and degree of ODE are two.
(iii) The order and degree of ODE are one.
(iv) The degree of $\frac{dy}{dx} + \sin(\frac{dy}{dx}) = 0$ is not defined as the differential equation is not a polynomial equation in its derivatives although it has order 1.
(v) The order is 2 and the degree of $\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx} + \sin y = 0$ is 1 as the differential equation is a polynomial equation in its derivatives although not a polynomial in y.
(vi) The order of $\left\{\frac{d^3y}{dx^3}\right\}^{\frac{3}{2}} + \left\{\frac{d^3y}{dx^3}\right\}^{\frac{2}{3}} = 0$ is 3. The L.C.M of the denominators of $\frac{3}{2}, \frac{2}{3}$ is 6. To find the degree, the said differential equation can be written as $\left\{\left(\frac{d^3y}{dx^3}\right)^{\frac{3}{2}}\right\}^6 = \left\{-\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}}\right\}^6$ i.e., $\left(\frac{d^3y}{dx^3}\right)^9 = \left(\frac{d^3y}{dx^3}\right)^4$. Hence the degree of the given differential equation is 9.

Remark: It may be mention here that the differential equation $\left(\frac{d^2y}{dx^2}\right)^4 = \left(\frac{d^2y}{dx^2}\right)^9$ can not be consider as $\left(\frac{d^2y}{dx^2}\right)^5 = 1$.

(vii) The order of $\left(\frac{d^2y}{dx^2}\right)^{-\frac{7}{2}}\frac{dy}{dx} + y\left(\frac{d^2y}{dx^2}\right)^{-\frac{5}{2}} = 0$ is 2. The power of highest order derivative is negative. But the degree of a differential equation is always positive. So to find the degree, we are multiplying $\left(\frac{d^2y}{dx^2}\right)^{\frac{7}{2}}$ in both side of the said differential equation and then we obtain $\frac{dy}{dx} + y\frac{d^2y}{dx^2} = 0$. Hence the degree of the given differential equation is 1.

Hypothesis(H): Let $\mathfrak{I} \subseteq \mathfrak{R}^n$ be a domain and $\mathbb{I} \subseteq \mathfrak{R}$ be an interval. Let $f : \mathbb{I} \times \mathfrak{I} \to \mathfrak{R}$ be a continuous function defined by $(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}) \mapsto f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^{n-1}y}{dx^{n-1}})$.

Definition 1.6 (**ODE in Normal form**) Assume Hypothesis(H) on f. An ordinary differential equation of order n is said to be normal form if

$$\frac{d^ny}{dx^n} = f\Big(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\Big). \tag{1.17}$$

Hypothesis(H_s): Let $\mathfrak{I} \subseteq \mathfrak{R}^n$ be a domain and $\mathbb{I} \subseteq \mathfrak{R}$ be an interval. Let $f : \mathbb{I} \times \mathfrak{I} \to \mathfrak{R}^n$ be a continuous function defined by $(x, \mathbf{y}) \mapsto f(x, \mathbf{y})$ where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$.

Definition 1.7 (**System of ODEs**) Assume Hypothesis(H_s) on **f**. A first order system of n ordinary differential equations is given by

$$\frac{d\mathbf{y}}{dx} = f(x, \mathbf{y}). \tag{1.18}$$

1.5 Types of Differential Equations

Differential equations may be classified in several ways. In this section we note that the independent variable may be implicit or explicit, and that higher order derivatives may appear.

1.5.1 An autonomous and non-autonomous differential equations

Definition 1.8 (**An autonomous and non-autonomous differential equations**)An autonomous differential equation is given by

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(\mathbf{y}, \lambda), \quad \mathbf{y} \in \Re^n, \quad \lambda \in \Re^k; \tag{1.19}$$

that is, the function $\mathbf{f}$ does not depend explicitly on the independent variable. If the function $\mathbf{f}$ does depend explicitly on x, then the corresponding differential equation is called non-autonomous.

In physical applications, we often encounter equations containing second, third, or higher order derivatives with respect to the independent variable. These are called second order differential equations, third order differential equations, and so on, where the the order of the equation refers to the *order* of the highest order derivative with respect to the independent variable that appears explicitly. In this hierarchy, a differential equation is called a first order differential equation.

Recall that Newtons second lawthe rate of change of the linear momentum acting on a body is equal to the sum of the forces acting on the bodyinvolves the second derivative of the position of the body with respect to time. Thus, in many physical applications the most common differential equations used as mathematical models are second order differential equations.

$$\frac{d^2\mathbf{y}}{dx^2} = m\mathbf{f} \tag{1.20}$$

Put $\mathbf{u} = \frac{d\mathbf{y}}{dx}$ in (1.20) and then it becomes to a autonomous differential equation

$$\frac{d\mathbf{u}}{dx} = m\mathbf{f} \tag{1.21}$$

1.5.2 Linear and Non-linear Differential Equations

Definition 1.9 (**Linear Differential Equation**) A differential equation (an ordinary or partially differential equation) which contains the dependent variable and its derivative as a first degree terms and no such term which is the product of the dependent variable (or its function) and its derivatives or not any transcendental function of the dependent variable will be called a **linear differential equation**. The general form of a linear differential equation is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = R(x)$$

where $a_0, a_1, a_2, \cdots, a_n, R, y$ are functions of an independent variable x.

Definition 1.10 (**Non-linear Differential Equation**) A differential equation (an ordinary and partially differential equations) which is not linear is called a **non-linear differential equation**.

So, the coefficients of linear differential equation are therefore either constant or functions of the independent variable or variables. As for examples, the differential equations (1.2), (1.3), (1.4), (1.6), (1.9), (1.11), (1.12), (1.13) and (1.14) are linear differential equations and the differential equations (1.1)(because of the term $y\frac{dy}{dx}$), (1.5)(because of the term $(x+y)^3\frac{dy}{dx}$), (1.7) and (1.8)(because of the term $\left(\frac{d^2y}{dx^2}\right)^2$), (1.10)(because of the term $x^3 \sin y$), (1.15) (because of the term $k\left(\frac{\partial^3 u}{\partial x^3}\right)^2$) are non-linear differential equations.

Example 1.2 Show that $(x+y)^2\frac{dy}{dx} + 5y = 3x^4$ is a non-linear ODE.

Proof. Since the co-efficient of $\frac{dy}{dx}$ is a function of x and y. So the differential equation is non-linear ODE.

Example 1.3 Determine the linearity of the following ODEs.
(i) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y^2 = 0$ C.H-96
(ii) $\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0.$

Solution. (i) Non-linear, since $\frac{dy}{dx}$ is multiplied by y and the last term is y^2 and not y.
(ii) Non-linear, since $\frac{d^2y}{dx^2}$ is of power two.

1.5.3 Homogeneous and Non-homogeneous Differential equations

Definition 1.11 (**Homogeneous Differential equation**) An ordinary differential equation is said to be homogeneous if there is no isolated constant term in the equation, i.e. if all the terms are proportional to a derivative of dependent variable (or dependent variable itself) and there is no term that contains a function of independent variable or constant alone.
A partial differential equation is said to be homogeneous if derivatives involved in it are of the same order.

A $n-th$ order linear differential equation of the form

$$P_0\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_ny = R \tag{1.22}$$

where y is the dependent variable, x is the independent variable and $P_0(\neq 0), P_1, P_2, \cdots, P_n$ and R are either constants or functions of x. If $R = 0$, then (1.22) is called a **homogeneous linear differential equation**.
In particular a second order homogeneous linear differential equation is given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \tag{1.23}$$

where P, Q are either constants or functions of x.

Remarks: A homogeneous differential equations has several distinct meanings:
(i) A first-order ordinary differential equation of the form $\frac{dy}{dx} = f(\frac{y}{x})$ is of the type of homogeneous equation.
(ii) A differential equation is said to be homogeneous if it has zero as a solution otherwise it is nonhomogeneous.

Definition 1.12 (**Non-Homogeneous Differential equation**) A differential equation which is not homogeneous is called non-homogeneous differential equation.

If $R \neq 0$, then (1.22) is called a **non-homogeneous linear differential equation**.

1.6 Genesis

The differential equations originate from physical phenomena are well known. In what follows, we a few problems and many more will be cited in course of the development of the subject. We however begin with an algebraic problem.

1.6.1 Algebraic Origin

The differential equations are formed by eliminating all the arbitrary constants that are involved in the relation between the dependent and independent variables in algebra. Depending on the number of arbitrary constants involve in the given equation differentiate it as many number of times successively. Then the elimination of the arbitrary constants from the resulting equations gives the required differential equation whose order is equal to the number of constants as the following problem illustrates.

Example 1.4 Find the differential equation from the relation $y = ax^2+a^2$ where a is an arbitrary constant.

Solution: The relation is given by

$$y = ax^2 + a^2 \tag{1.24}$$

The relation (1.24) contain only one arbitrary constant i.e. a, so order of the differential equation is of first order.

Differentiating (1.24) with respect to x, we get

$$\frac{dy}{dx} = 2xa \qquad \Rightarrow a = \frac{1}{2x}\frac{dy}{dx}$$

Substituting the value of a in (1.24), we get

$$y = \frac{1}{2x}\frac{dy}{dx}x^2 + \left(\frac{1}{2x}\frac{dy}{dx}\right)^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 + 2x^3\frac{dy}{dx} - 4x^2y = 0.$$

Which is the required differential equation.

Example 1.5 Find the differential equation from the relation

$$ax^2 + by^2 = 1, \tag{1.25}$$

where a and b are arbitrary constants.

Solution: The relation (1.25) contain two arbitrary constants i.e. a and b, so order of the differential equation is of second order. Differentiating (1.25) *w.r.t.* x, we get,

$$2ax + 2by\frac{dy}{dx} = 0 \tag{1.26}$$

Differentiating again with respect to x, we get,

$$2a + 2b\left(\frac{dy}{dx}\right)^2 + 2by\frac{d^2y}{dx^2} = 0 \tag{1.27}$$

From (1.26) we get,

$$\frac{a}{b} = -\frac{y}{x}\frac{dy}{dx} \tag{1.28}$$

Now from (1.27) and (1.28), we get,

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = \frac{y}{x}\frac{dy}{dx} \Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$$

which is the required ordinary differential equation.

1.6.2 Geometric Origin

The following is a nice situation in geometry which gives birth to a differential equation.

Example 1.6 Find the differential equation of all family of curves for which the length of the part of the tangent between the point of contact (x, y) and the y-axis is equal to the y-intercept of the tangent.

Solution: The equation of the tangent to any point $P(x, y)$ of the curve $y = f(x)$ (cf. Figure-1.1) is given by

$$Y - y = \frac{dy}{dx}(X - x) \Rightarrow Y - X\frac{dy}{dx} = y - x\frac{dy}{dx} \Rightarrow \frac{X}{x - y\frac{dx}{dy}} + \frac{Y}{y - x\frac{dy}{dx}} = 1 \tag{1.29}$$

Then the y intercept of the tangent is $(y - x\frac{dy}{dx})$ and the length of the tangent between the point (x, y) and y-axis is the length $PQ = \sqrt{PR^2 + QR^2} = \sqrt{(x\tan\Psi)^2 + x^2} = \sqrt{x^2(\frac{dy}{dx})^2 + x^2} = x\sqrt{1 + (\frac{dy}{dx})^2}$.

Then we have, $x\sqrt{1 + (\frac{dy}{dx})^2} = y - x\frac{dy}{dx} \Rightarrow x^2[1 + (\frac{dy}{dx})^2] = (y - x\frac{dy}{dx})^2 \Rightarrow x^2 = y^2 - 2xy\frac{dy}{dx}$.

Example 1.7 Find the differential of all curves in the plane the tangent at every point of which is parallel to the line joining the origin to that point.

Solution: Let $P(x,\ y)$ be a point on the curve. The slope of the line joining the origin O to P is $\frac{y}{x}$. By the given condition, we then get $\frac{dy}{dx} = \frac{y}{x}$.

1.6.3 Mechanical Origin

Often differential equation describes a motion in mechanics as will be clear from the following equation.

Example 1.8 A particle moves in a straight line such that its acceleration at a point is proportional to its displacement measured from a fixed point on the line. Describe the motion.

Solution: Let x denoted the displacement of the particle. Then by the given condition, we get the equation of motion as $\frac{d^2y}{dx^2} = kx$. Clearly, the motion is simple harmonic if $k < 0$.

1.6.4 Physical Origin

The following is a nice situation in physics which gives birth to a differential equation.

Example 1.9 Newtons second law: the rate of change of the linear momentum acting on a body is equal to the sum of the forces acting on the bodyinvolves the second derivative of the position of the body with respect to time. Thus, in many physical applications the most common differential equations used as mathematical models are second order differential equations.

$$\frac{d^2x}{dt^2} = mf \tag{1.30}$$

1.6.5 Chemical Origin

The following is a situation in chemistry which is best described by a differential equation.

Example 1.10 If hydriodic acid HI decomposes at a rate proportional to $(1 - x)^2$ where x denoted the quantity of hydriodic acid at time t and if this decomposition is retarded by a quantity proportional to x^2, describe how the acid decomposes.

Solution: Let x denotes the quantity of hydriodic acid at time t. Then $\frac{dx}{dt} = k_1(1 - x)^2 - k_2x^2$ which describes the rate of decomposition.

1.6.6 Population Origin

The following is a nice situation in biology which gives birth to a differential equation.

Example 1.11 Let us consider the following situation of fish: $X(t)$ be the biomass of fish, $\dot{X}(t)$ be the growth rate of X with respect to time t, $N_x(t)$ be the amount of nutrients at time t, r_x be the natural growth rate which is independent of supplied nutrients N_x, θ_x be the constant deterioration rate, L_x be the environmental carrying capacity and h_x be the harvesting rates. The find the differential equation the of the said fish.

Solution: The governing dynamical system of fish is as follows:

$$\dot{X}(t) = r_x\left(\frac{N_x}{k_x + N_x}\right)\left(1 - \frac{X}{L_x}\right)X - h_x,, \text{ where } k_x \text{ is the positive constant.}$$

1.6.7 Economical Origin

The following is a situation in Econometric which is best described by a differential equation.

Example 1.12 Let us consider the rate of instantaneous change of the price $P(t)$ is directly proportional to the difference in the demand D and the supply S of this product for each time t. If the demand and supply depend on the price as well as the time t, set the problem in precise model.

Solution: Here $P(t)$ denotes the price at time t. Let $D(t, p)$ and $S(t, p)$ denote the demand and the supply at time t when the price is $P(t)$. Hence the instantaneous change of the price $P(t)$ at time t is given by

$$\frac{dP}{dt} = k\{D(t, P) - S(t, P)\} \text{ where } k \text{ is a positive constant.}$$

1.6.8 Biological Origin

The following example is drawn from Biology to demonstrate the need of differential equation in describing many phenomena.

Example 1.13 Bacteria are produced at a rate proportional to their available quantity but at the same time they generate poison destroying them at a rate proportional to the amount of the poison and the quantity of bacteria, the rate of generation of poison being proportional to the available quantity of bacteria. Determine the number of bacteria produced in time t.

Solution: Let $P(t)$ be the amount of poison in time t, $N(t)$ the number of bacteria at time t. Assuming P and N to be differentiable functions of t, we can describe the above phenomenon as

$$\frac{dN(t)}{dt} = aN(t) - bN(t)P(t), \ \frac{dP}{dt} = cN(t), \text{ where } a, b, c \text{ are constants.}$$

1.7 Initial-value problems

Definition 1.13 (**Initial Value Problem for an ODE**) An initial-value problem(IVP) is a differential equation together with subsidiary conditions to be satisfied by the solution function and its derivatives, all given at the same value of the independent variable.
Let $x_0 \in \mathbb{I}$ and $(y_1, y_2, \cdots, y_n) \in \mathfrak{I}$ be given. An IVP for an ODE in normal form is a relation satisfied by an unknown function y given by

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right), \ \left(\frac{d^r y}{dx^r}\right)_{x=x_0} = y_r, \ r = 0, 1, 2, \cdots, n-1. \tag{1.31}$$

A second order IVP may in general be put in the standard form as.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \tag{1.32}$$

where P,Q and X are functions of x. Subject to the conditions

$$y(a) = c_1 \qquad \text{and} \qquad \left(\frac{dy}{dx}\right)_{x=a} = c_2, \tag{1.33}$$

where a is a specific value of the independent variable x and c_1, c_2 are two constants. Hence a solution to an initial-value problems is to find a $y(x)$ that satisfies the differential equation (1.32) as well as the given initial condition (1.33). If particular, $X = 0$ and $c_1 = c_2 = 0$, the problem is said to be a homogeneous initial-value problem.

Example 1.14 Consider the differential equation $\frac{d^2y}{dx^2} + 4y = 0$; $y(0) = 0$, $\left(\frac{dy}{dx}\right)_{x=0} = 2$. This problem consists in finding a solution of the differential equation which assumes the value 0 at $x = 0$ and whose first derivative assumes the value 2 at $x = 0$. Thus this is an initial-value problem.

1.8 Boundary-value problems

Definition 1.14 (**Boundary-value problems for an ODE**) A boundary value problem(BVP) is a differential equation together with subsidiary conditions to be satisfied by the solution function and its derivations, where the conditions are given for more than one values of the independent variable.

A n order boundary-value problem in linear homogeneous differential equation may in general be put in the form:

$$\text{ODE:} \qquad p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{(n-1)}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0 \tag{1.34}$$

$$\begin{aligned}\text{BCS:} \qquad & \alpha_k y(a) + \alpha_k^{(1)}\frac{dy}{dx}\bigg|_{x=a} + \cdots + \alpha_k^{(n-1)}\frac{d^{(n-1)}y}{dx^{n-1}}\bigg|_{x=a} + \beta_k y(b) + \beta_k^{(1)}\frac{dy}{dx}\bigg|_{x=b} + \cdots \\ & + \beta_k^{(n-1)}\frac{d^{(n-1)}y}{dx^{n-1}}\bigg|_{x=b} = 0, \qquad k = 1, 2, \cdots, n.\end{aligned} \tag{1.35}$$

where the functions $p_0(x), p_1(x), \cdots, p_n(x)$ are continuous on $[a, b]$, $p_0(x) \neq 0$ on $[a, b]$. Also assume that $a \neq b$, $\alpha_k, \alpha_k^{(1)}, \cdots, \alpha_k^{(n-1)}, \beta_k, \beta_k^{(1)}, \cdots, \beta_k^{(n-1)}$ are all real constants and at least one of $\alpha_k, \alpha_k^{(1)}, \cdots, \alpha_k^{(n-1)}, \beta_k, \beta_k^{(1)}, \cdots, \beta_k^{(n-1)}$ is non zero for all $k = 1, 2, \cdots, n$.

Also, a second order boundary-value problem in linear differential equation may be put in the form:

$$\frac{d^2y(x)}{dx^2} + P(x)\frac{dy(x)}{dx} + Q(x)y(x) = R(x), \tag{1.36}$$

with the boundary conditions

$$A_1y(a) + B_1\Big(\frac{dy}{dx}\Big)_{x=a} = c_1,\ A_2y(b) + B_2\Big(\frac{dy}{dx}\Big)_{x=b} = c_2, \tag{1.37}$$

where the functions P, Q, R are continuous $[a, b]$ and $A_1, B_1, c_1, A_2, B_2, c_2$ are all real constants. Also assume that $a \neq b$, A_1 and B_1 are not zero at a time and similarly A_2 and B_2 are also not all zero at a time.

If the differential equation as well as the boundary condition are all homogeneous, that is, if $R = 0, c_1 = c_2 = 0$, then this problem is said to be a homogeneous boundary-value problems. Hence, a solution to a boundary value problem is to find a $y(x)$ that satisfies the differential equation (1.36) as well as the given boundary condition (1.37).

Example 1.15 Consider the differential equation $\frac{d^2y}{dx^2}+y = 5;\ y(0) = 7, y(\frac{\pi}{2}) = 16$. This problem consists in finding a solution of the differential equation which assumes the values 7 at $x = 0$ and whose first derivative assumes the value 16 at $x = \frac{\pi}{2}$. Both of these conditions related to one value of x, namely $x = 0$. That is, the conditions related to the two different values of x, 0 and $\frac{\pi}{2}$. Thus this is an boundary-value problem.

1.9 Solution of Differential Equations

A function is said to be a **solution** of a differential equation, over a particular domain of the independent variable, if its substitution into the equation reduces that equation to an identity everywhere within that domain.

For an $n-$th-order ordinary differential equation (1.16), a function $y = \phi(x)$, which is n times differentiable and satisfies the differential equation in some interval $\mathbb{I}$ when substituted into the equation, is called a solution of the differential equation over the interval $\mathbb{I}$.

Definition 1.15 (**Solution of an ODE**) A function ϕ is said to be a solution of ODE (1.16) if $\phi \in \mathbb{C}^n(\mathbb{I})$ and

$$F\Big(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \cdots, \phi^{(n)}(x)\Big) = 0,\quad \forall x \in \mathbb{I} \tag{1.38}$$

where $\phi^{(n)}$ stands for n^{th} derivative of the function $x \mapsto \phi(x)$ with respect to the independent variable x.

Definition 1.16 (**Solution of an ODE in Normal form**) A function $\phi \in \mathbb{C}^n(\mathbb{I}_0)))$ where $\mathbb{I}_0 \subseteq \mathbb{I}$ is a subinterval, is called a solution of ODEs (1.17) if for every $x \in \mathbb{I}_0$, the $(n + 1)-$tuple $(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \cdots, \phi^{(n-1)}(x)) \in \mathbb{I} \times \Im$ and

$$\phi^{(n)}(x) = f\Big(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \cdots, \phi^{(n-1)}(x)\Big),\quad \forall x \in \mathbb{I}_0. \tag{1.39}$$

Definition 1.17 (**Solution of an IVP for an ODE**) A solution ϕ of ODE (1.17) is said to be a solution of IVP if $x_0 \in \mathbb{I}_0$ and

$$\Big(\frac{d^r\phi}{dx^r}\Big)_{x_0} = y_r,\ r = 0, 1, 2, \cdots, n-1.$$

This solution is denoted by $\phi(\cdot; x_0, y_0, y_1, \cdots, y_{n-1})$ to remember the IVP solved by ϕ.

Definition 1.18 (**Local and Global solutions of an IVP**) Let ϕ be a solution of an IVP for ODE (1.17) according to Definition 1.13.

1 If $\mathbb{I}_0 \subseteq \mathbb{I}$, then ϕ is called a **local solution** of IVP.

2 If $\mathbb{I}_0 = \mathbb{I}$, then ϕ is called a **global solution** of IVP.

Remarks:

1 Note that in all our definitions of solutions, a solution always comes with its domain of definition. Sometimes it may be possible to extend the given solution to a bigger domain. We address this issue in the next chapters.

2 When $n = 1$, geometrically speaking, graph of solution of an IVP is a curve passing through the point (x_0, y_0).

Definition 1.19 (General Solution) The solution of a differential equation is called its general solution if its contains a number of arbitrary constants equal to the order of the differential equation. This solution is also called a **complete solution** or a **complete primitive** or a **complete integral**.

As for example $y = A\, cos\, x + B\, sin\, x$ is the complete solution of a differential equation $\frac{d^2y}{dx^2} + y = 0$, since it contains two arbitrary constants A and B as well as the order of the differential is also two.

Definition 1.20 (Particular Solution) A solution of a differential equation by giving particular values to the arbitrary constants in its general solution is called a particular solution of that equation.

As for example, if we put $A = 1$ and $B = 0$ in the general solution $y = A\, cos\, x + B\, sin\, x$ of the differential equation $\frac{d^2y}{dx^2} + y = 0$, then $y = cos\, x$ is the particular solution of this equation.

Definition 1.21 (Singular Solution) Sometime, the general solution of any differential equation does not include all possible solutions of the differential equation. In otherworld, there may exist such a solution which can not be obtained by giving any particular values to those arbitrary constants to the general solution. This is called a singular solution of that differential equation.

As for example $y = c\,x + \frac{a}{c}$ is the general solution of the differential equation $y = p\,x + \frac{a}{p}$ where $p \equiv \frac{dy}{dx}$. But it is seen that $y^2 = 4ax$ is also a solution of this differential equation, but it cannot be obtained by giving any particular value of c to its general solution. So $y^2 = 4ax$ is a singular solution of the differential equation $y = p\,x + \frac{a}{p}$.

In some differential equations, the general solution consists of terms involving the arbitrary constants and terms giving a function of the independent variable. The first part is called the **complementary function** and the remaining part, which can be obtained by giving the value zero to each of the arbitrary constants, is called the **particular integral**. If the general solution is given by

$$y = A\, cosx + B\, sinx + xe^x,$$

where A and B are arbitrary constants, then the complementary function is $A\, cosx + B\, sinx$ and particular integral is xe^x.

1.10 Geometric interpretation of a first order ODE and its solution

We now define some terminology that we use while giving a geometric meaning of an ODE given by

$$\frac{dy}{dx} = f(x, y) \tag{1.40}$$

We recall that f is defined on D in $\Re^2$. In fact, $D = \mathbb{I} \times \mathbb{J}$ where $\mathbb{I}, \mathbb{J}$ are sub-intervals of $\Re$.

Definition 1.22 (**Line element**) A line element associated to a point $(x, y) \in D$ is a line passing through the point (x, y) with slope p. We use the triple (x, y, p) to denote a line element.

Definition 1.23 (**Direction field/ Vector field**) A direction field (sometimes called vector field) associated to the ODE (1.40) is collection of all line elements in the domain D where slope of the line element associated to the point (x, y) has slope equal to $f(x, y)$. In other words, a direction field is the collection $\left\{(x, y, f(x, y)) : (x, y) \in D\right\}$.

Remark (Interpretations):

1 The ODE (1.40) can be thought of prescribing line elements in the domain D.

2 Solving an ODE can be geometrically interpreted as finding curves in D that fit the direction field prescribed by the ODE. A solution (say ϕ) of the ODE passing through a point $(x_0, y_0) \in D$ (i.e., $\phi(x_0) = y_0$) must satisfy $\phi'(x_0) = f(x_0, y_0)$. In other words,

$$(x_0, y_0, \phi'(x_0)) = (x_0, y_0, f(x_0, y_0))$$

3 That is, the ODE prescribes the slope of the tangent to the graph of any solution (which is equal to $\phi'(x_0)$). This can be seen by looking at the graph of a solution.

4 Drawing direction field corresponding to a given ODE and fitting some curve to it will end up in finding a solution, at least, graphically. However note that it may be possible to fit more than one curve passing through some points in D, which is the case where there are more than one solution to ODE around those points. Thus this activity (of drawing and fitting curves) helps to get a rough idea of nature of solutions of ODE.

5 A big challenge is to draw direction field for a given ODE. One good starting point is to identify all the points in domain D at which line element has the same slope and it is easy to draw all these lines. These are called isoclines; the word means leaning equally.

1.11 Worked Out Examples

Example 1.16 Determine the order and degree of the following differential equation

(i)$(\frac{dy}{dx})^2 + 3y^2 = 0$ (ii) $\left(\frac{d^2y}{dx^2}\right)^2 + xy = \frac{dy}{dx}$ (iii) $\sqrt{\frac{dy}{dx}} = 2y$.

(iv) $\left(\frac{dy}{dx}\right)^{\frac{2}{3}} = 3 + \frac{d^2y}{dx^2}$ (v)$\left(\frac{d^2y}{dx^2} + 1\right)^{\frac{3}{2}} = 3x\frac{dy}{dx}$ (vi) $y + \frac{dy}{dx} = e^{\frac{d^2y}{dx^2}}$

Solution: (i) Order is 1 and degree is 2.

(ii) Order is 2 and degree is 2.

(iii) $\sqrt{\frac{dy}{dx}} = 2y \Rightarrow \frac{dy}{dx} = 4y^2$, order is 1 and degree is 1.

(iv) $\left(\frac{dy}{dx}\right)^{\frac{2}{3}} = 3 + \frac{d^2y}{dx^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(3 + \frac{d^2y}{dx^2}\right)^3$.
Hence the order is 2 and degree is 3.
(v) $\left(\frac{d^2y}{dx^2} + 1\right)^{\frac{3}{2}} = 3x\frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2} + 1\right)^3 = 9x^2\left(\frac{dy}{dx}\right)^2$.
Hence the order is 2 and degree is 3.
(vi) The differential equation can be written as $\frac{d^2y}{dx^2} = \log(y + \frac{dy}{dx})$, so the degree of the said differential equation can not be defined as it is not a polynomial of derivatives although it has order 2.

Example 1.17 Show that the differential equation of the family of circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{1.41}$$

(where g, f, c are parameters) is $\left\{1 + (\frac{dy}{dx})^2\right\}\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$.

Solution: Equation (1.41) contains three parameter g, f and c. Differentiating (1.41) with respect to x , we get $2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$. Again differentiating with respect to x, we get

$$2 + 2y\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^2 + 2f\frac{d^2y}{dx^2} = 0 \tag{1.42}$$

Differentiating again with respect to x, we get,

$$2y\frac{d^3y}{dx^3} + 6\frac{dy}{dx}\frac{d^2y}{dx^2} + 2f\frac{d^3y}{dx^3} = 0. \tag{1.43}$$

Now from (1.43) we get,

$$f = -\frac{(y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2})}{\frac{d^3y}{dx^3}} \tag{1.44}$$

From (1.42) & (1.44), the required result is $\left\{1 + (\frac{dy}{dx})^2\right\}\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$.

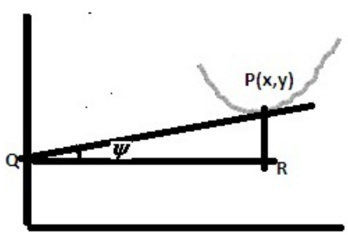

Figure 1.1: Family of Curves

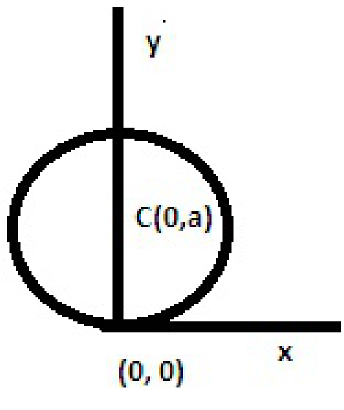

Figure 1.2: Circles

Example 1.18 Obtain the differential equation of all circles each of which touches the x axis at the origin. C.U(Hons.)-85, 93; V.U.(Hons.)-88, 97; IAS(Prel.)-99

Solution: The equation of the circles touching the x-axis at the origin (cf. Figure-1.2) is

$$(x-0)^2+(y-a)^2=a^2 \Rightarrow x^2+y^2-2ay=0, \tag{1.45}$$

where a is an arbitrary constant. Differentiating both sides $w.r.t.$ x, we get,

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0,\ a=\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}.$$

Putting the value of a in the equation (1.45), we get,

$$x^2+y^2-2\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}y=0 \Rightarrow (x^2+y^2)\frac{dy}{dx}-2y^2\frac{dy}{dx}=2xy \ \Rightarrow (x^2-y^2)\frac{dy}{dx}=2xy.$$

Example 1.19 Find the differential equation of all family of circles having their centres on the $y-$ axis.

Solution: The equation of the family of circles having their centres on the y-axis to be

$$x^2+(y-a)^2=r^2, \tag{1.46}$$

where $a,\ r$ are arbitrary constants. This is a two parameters family of curves. To find the differential equation of the family of curves (1.46), we have to eliminate both a and r. Differentiating (1.46)with respect to x, we get

$$2x+2(y-a)\frac{dy}{dx}=0 \ \Rightarrow \ x+(y-a)\frac{dy}{dx}=0. \tag{1.47}$$

Expressing (1.47) in the form, $\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}=a$ and then differentiating the above relation $w.r.t.$ x, we find that

$$\frac{\frac{dy}{dx}\left(1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-\frac{d^2y}{dx^2}\left(x+y\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)^2}=0$$

or, $\quad x\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0$ is the differential equation of the family of circles.

Example 1.20 Obtain the differential equation corresponding to the primitive:

$$(x-\alpha)^2+(y-\beta)^2=r^2 \tag{1.48}$$

where r is a fixed constant and α, β are arbitrary constants. Give a geometrical interpretation of the result. **B.U.(Hons.) 1982**

Solution: Differentiating (1.48) $w.r.t.$ x, we obtain

$$(x-\alpha)+(y-\beta)\frac{dy}{dx}=0 \tag{1.49}$$

Differentiating (1.49) again $w.r.t.$ x, we obtain

$$1+(\frac{dy}{dx})^2+(y-\beta)\frac{d^2y}{dx^2}=0 \tag{1.50}$$

From (1.50) we get, $y-\beta=-\frac{1+(\frac{dy}{dx})^2}{\frac{d^2y}{dx^2}}$ and hence (1.49) gives $x-\alpha=\frac{\frac{dy}{dx}(1+(\frac{dy}{dx})^2)}{\frac{d^2y}{dx^2}}$. These values when substituted in (1.48) leads to the required equation

$$r^2\left(\frac{d^2y}{dx^2}\right)^2=\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 \tag{1.51}$$

Geometrical interpretation. If we vary α and β in (1.48) we get a system of circles of given radius r, having their centres anywhere in the $xy-$ plane. The differential equation (1.51) expresses the fact that for every member of the system the radius of curvature has everywhere the constant value r.

Example 1.21 The equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \qquad \textbf{WBSSC 2001} \tag{1.52}$$

(where a and b are fixed constants and λ is an arbitrary parameter which can assume all real values) represents a family of confocal conics. To obtain the differential equation of this family.

Solution: The required differential equation is obtained by eliminating λ from (1.52) and the derived equation

$$\begin{aligned}
& \frac{2x}{a^2+\lambda}+\frac{2yy^{'}}{b^2+\lambda}=0,\ (y^{'}=\frac{dy}{dx})\\
\Rightarrow\quad & \frac{x}{a^2+\lambda}=-\frac{yy^{'}}{b^2+\lambda}=\frac{x+yy'}{a^2-b^2}\\
\Rightarrow\quad & \frac{x^2}{a^2+\lambda}=\frac{x(x+yy')}{a^2-b^2} \quad \text{and} \quad \frac{y^2}{b^2+\lambda}=-\frac{y}{y'}\cdot\frac{x+yy'}{a^2-b^2}
\end{aligned} \tag{1.53}$$

Then using (1.52) and (1.53), we have the required differential equation as

$$(a^2-b^2)y^{'}=(xy^{'}-y)(x+yy^{'})$$

Example 1.22 State with reasons whether the following differential equations are linear:

$(i)\ \frac{d^2y}{dx^2}+y\frac{dy}{dx}+y^2=0$ $\qquad$ $(ii)\ x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=1-\log x,\ x>0$

$(iii)\ x\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}-\sin x\sqrt{y}=0,\ y>0$ $\qquad$ $(iv)\ \frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+x\sin y=0$

Solution: (i) The non-linear terms of the differential equation are $y\frac{dy}{dx}$ and y^2. So the differential equation is non-linear.
(ii) Each terms of the differential equation is in linear form. So the differential equation is linear.
(iii) The non-linear term of the differential equation is $\sin x\sqrt{y}$. So the differential equation is non-linear.
(iv) The non-linear term of the differential equation is $x\sin y$. So the differential equation is non-linear.

1.12 Multiple Choice Questions(MCQ)

1. The type of the following differential equation $y'' + \sin(x + y) = \sin x$ is
(a) linear, homogeneous (b) nonlinear, homogeneous **Gate(MA): 2001**
(c) linear, nonhomogeneous (d) nonlinear, nonhomogeneous
Ans. (d) is correct.

2. If $y = \ln(\sin(x+a))+b$ where a and b are constants, is the primitive, then the corresponding lowest order differential equation is
(a) $y'' = -(1 + (y')^2)$ (b) $y'' = 1 + (y')^2$
(c) $y'' = -(2 + (y')^2)$ (d) $y'' = -(3 + (y')^2)$ [JAM CA-2005]
Ans. (a)
Hint. $y = \ln(\sin(x + a)) + b$ contains two arbitrary constants. Eliminating a and b, we get, $y'' = -(1 + (y')^2)$.

3. The differential equation representing all circles centrad at $(1, 0)$ is
(a) $x + y\frac{dy}{dx} = 1$ (b) $x - y\frac{dy}{dx} = 1$ (c) $y - x\frac{dy}{dx} = 1$ (d) $y + x\frac{dy}{dx} = 1$
Ans. (a) JAM CA-2010

4. The differential equation representing the family of circles touching y- axis at the origin is
(a) Non linear and of first order (b) linear and of second order JAM(MA):2006
(c) exact and linear but not homogeneous (d) exact, homogeneous and linear
Ans. (a)
Hint. Like the example 1.18.

5. The differential equation $(3y - 2x)\frac{dy}{dx} = 2y$ JAM CA-2006
(a) homogeneous but not linear (b) linear and homogeneous
(c) linear but not homogeneous (d) homogeneous and linear
Ans. (a)

6. The degree of $\frac{d^2y}{dx^2} = \log(y + \frac{dy}{dx})$ is
(a) 1 (b) 0 (c) Does not exist (d) 2
Ans. (c)
Hint. The R.H.S of the given differential equation can not be a polynomial of $\frac{dy}{dx}$.

7. The order and degree of $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}} = \left(y + \frac{dy}{dx}\right)^{\frac{1}{2}}$ are
(a) 1, 3 (b) 2, 1 (c) 2, Does not exist (d) 2, 2
Ans. (d)

8. The order and degree of $\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{3}{2}} = 0$ are
(a) 1, 3 (b) 4, 1 (c) 2, Does not exist (d) 3, 2 [IAS(Prel.) -2006;]
Ans. (b)
Hint. The problem is same with (vii) of Example 1.1.

1.13 Review Exercises

1 Define an ordinary differential equation. What do you mean by the degree and order of a differential equation?

2 Explain the terms: general solution, a particular solution, a singular solution as applied to an ordinary differential equation.

3 Find the differential equation, eliminate the arbitrary constants a, b, c from the equation $y = a + b\,e^{5x} + c\,e^{-7x}$. **Ans.** $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 35\frac{dy}{dx}$; C.U(Hons.) 1995

4 Find the differential equation, eliminate the arbitrary constants a, b from the equation $xy = ae^{x} + b\,e^{-x} + x^2$. **Ans.** $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$; IAS-1992

5 Find the differential equation, eliminate the arbitrary constants c from the equation $y = c(x-c)^2$. **Ans.** $(y')^3 = 4y(xy' - 2y)$ IAS(Prel.)-2009

6 Find the differential equation, eliminate the arbitrary constants a, b, c, d from the equation $y = \frac{ax+b}{cx+d}$. **Ans.** $3(\frac{d^2y}{dx^2})^2 = 2\frac{dy}{dx}\frac{d^3y}{dx^3}$.
Hint. The given equation can be written as $y = e\frac{x+f}{x+g}$ where $e = \frac{a}{c}$, $f = \frac{b}{a}$ and $g = \frac{d}{c}$. So the independent arbitrary constants are e, f, g. Eliminating the independent arbitrary constants e, f, g, we get the required differential equation $3(\frac{d^2y}{dx^2})^2 = 2\frac{dy}{dx}\frac{d^3y}{dx^3}$.

7 Find the order of differential equation, eliminate the arbitrary constants a, b, c, d from the equation $y = \frac{ax+b}{cx+d}$ with $c + d = 0$. **Ans.** 2.
Hint. The given equation can be written as $y = e\frac{x+f}{x-1}$ where $e = \frac{a}{c}$, $f = \frac{b}{a}$. So the independent arbitrary constants are e, f. As the number of independent arbitrary constants is 2, the the order of differential equation is 2.

8 Show that the substitution $x = sinh\, z$ transforms the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 4y$ into $\frac{d^2y}{dz^2} = 4y$. **C.U(Hons.) 1982**

9 Show that the differential equation of all parabolas having their axes parallel to the $y-$ axis is $\frac{d^3y}{dx^3} = 0$. **C.U(Hons.) 1994; N.B.U(Hons.) 2007**

10 Show that the differential equation of all parabolas with foci at the origin and axes along the $x-$ axis is $y(\frac{dy}{dx})^2 + 2x\frac{dy}{dx} - y = 0$. **V.U(Hons.) 2002**
Hint. Here the parabola is $(x+2a)^2 = x^2 + y^2$, where a is the parameter.

11 Obtain the differential equation of all circles each of which touches the y axis at the origin.
Ans. $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$; IAS(Prel.)-1997
Hint. Like the example 1.18.

12 Find the differential equation of all family of circles which pass through the origin and whose centres on the $x-$ axis. **Ans.** $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$; **B.U(Hons.)-1987, IAS(Prel.)-2002**.

13 Obtain the differential equation of all conics whose axes coincide with the axes of coordinates.
Hint. General equation of the conic is $ax^2 + by^2 = 1$.
Ans. $xy\frac{d^2y}{dx^2} + x(\frac{dy}{dx})^2 - y\frac{dy}{dx} = 0$.

14 Prove that the differential equation of the circles through the intersection of the circle $x^2 + y^2 = 1$ and the line $x - y = 0$ is
$(x^2 - 2xy - y^2 + 1)dx + (x^2 + 2xy - y^2 - 1)dy = 0$ **V.U(Hons.)-2017**
Hint. General equation of the circle is $x^2 + y^2 - 1 + \lambda(x - y) = 0$ where λ is a parameter.

15 The motion of a particle of mass m moving along a straight line is governed by the equation

$$mx'' + rx' + kx = 0, \text{ where } r \text{ and } k \text{ are constants. Verify that}$$
$$x = Ae^{-(\frac{r}{2m})t}\cos(\frac{\omega}{2m}t + B), \ (\omega^2 = 4mk - r^2)$$

satisfies the differential equation, A and B being arbitrary constraints.

16 Find the differential equations of the family of parabolas $y = cx^2$ when would the equation be meaningful? **Ans.** $xy' = 2y$. Everywhere except when c is indeterminate when $x = 0,\ y = 0$.

17 Determine the order and degree of the following ODEs.
(i)$\frac{d^2y}{dx^2} + \sin(\frac{dy}{dx}) = 0$. **Ans.** Order is 2 and degree does not exist.
(ii)$e^{\frac{d^3y}{dx^3}} + x + \frac{dy}{dx} = 0$. **Ans.** Order is 3 and degree does not exist.
(iii)$\left\{\frac{d^2y}{dx^2}\right\}^{\frac{1}{2}} + \left\{\frac{d^2y}{dx^2}\right\}^{\frac{2}{5}} = 0$. **Ans.** Order is 2 and degree is 5.
(iv)$\left(\frac{d^2y}{dx^2}\right)^{-\frac{3}{2}}\frac{dy}{dx} + y\left(\frac{d^2y}{dx^2}\right)^{-\frac{7}{2}} = 0$. **Ans.** Order is 2 and degree is 2.

18 Show that the substitution $x = e^z$ transforms the equation
$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = 0$ into $\frac{d^2y}{dz^2} + 4y = 0$.

19 Show that the substitution $z = sin\,x$ transforms the equation
$\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + y\cos^2 x = 0$ into $\frac{d^2y}{dz^2} + y = 0$. **C.U(Hons.) 1983**

20 Construct a differential equation by elimination of the arbitrary constants a and b from each of the relations
(i) $ax^2 + by^2 = 1$ **Ans.** $xyy'' + x(y')^2 - yy' = 0$
(ii) $(x-a)^2 + (y-b)^2 = r^2$ **Ans.** $r^2(y'')^2 = (1 + (y')^2)^3$
(iii) $x = a\cos nt + b\sin nt$. **Ans.** $\frac{d^2x}{dt^2} + n^2x = 0$

21 Eliminating α from the following equations. State the order and degree of the resulting differential equations; say whether linear or not
(i) $x^2 + y^2 - 2\alpha y = \alpha^2$
(ii) $y = \alpha x + \alpha - \alpha^3$
(iii) $x\cos\alpha + y\sin\alpha = a$.
Ans. (i) $x^2(y')^2 - 4xyy' - x^2 - 2y^2(y')^2 = 0$ (First order, second degree and non-linear)
(ii)$y = xy' + y' - (y')^3$ (First order, third degree and non-linear)
(iii) $(y - xy')^2 = a^2(1 + (y')^2)$ (First order, second degree and non-linear)

22 Show that $(x+y)\frac{dy}{dx} + 5y^2 = 3x^4$ is a non-linear ODE.

23 Determine the nature of the following ODEs.
(i)$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ **Ans.** Linear and homogenous; C.H-96.
(ii)$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^2$ **Ans.** Linear and non-homogenous.

24 Show that the differential equation representing the family of all straight lines which have an intercept of constant length L between the coordinate axes is $x\frac{dy}{dx} - y = \frac{L\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}}$
JAM(MA)-2008

25 Show that the adiabatic law $pv^\gamma = c$ is a fixed and c is an arbitrary constant, lead to $v\frac{dp}{dv} + \gamma p = 0$.

26 Show that the substitution $x = e^u$ transforms the equation
$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \cos x$ into $\frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \cos x$. JAM(MA)–2010

Chapter 2

Stability Analysis of Differential Equations

2.1 Introduction

The main purpose of developing stability theory is to examine the dynamic responses of a system to disturbances as time approaches infinity. It has been and still is the subject of intense investigations due to its intrinsic interest and its relevance to all practical systems in engineering, natural science, social science and finance. Lyapunov stability theory is the foundation of stability analysis for dynamic systems that are mathematically described by ordinary differential equations (ODE). Also, Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point. We have well established tools to analyze and stabilize linear systems.

The evolution of stability theory has been very rapid and extensive. The present chapter comes mainly from my research results and teaching of graduate students in the stability of dynamical systems. In this chapter, we define various stabilities mathematically, illustrate their relations using examples and discuss the main mathematical tools for stability analysis. The stability of linear systems with constant coefficients / variable coefficients and homogeneous / non-homogeneous systems are also introduced in details. The necessary and sufficient conditions are systematically developed for stability, uniform stability, uniformly asymptotic stability and instability.

In dynamical systems, an orbit is called Lyapunov stable if the forward orbit of any point is in a small enough neighborhood or it stays in a small (but perhaps, larger) neighborhood. Various criteria have been developed to prove stability or instability of an orbit. Under favorable circumstances, the question may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions. In practice, any one of a number of different stability criteria are applied.
In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter

values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets changes.
In this chapter, we have discussed the overall concepts about the stability theory and bifurcation theory.

2.2 Dynamical System

A dynamical system is a concept in mathematics where a fixed rule describes how a point in a geometrical space depends on time. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each springtime in a lake, etc. Specially, the dynamical system has a mathematical form of system of ODEs. So we have,

Hypothesis(H_s): Let $\mathfrak{I} \subseteq \mathfrak{R}^n$ be a domain and $\mathbb{I} \subseteq \mathfrak{R}$ be an interval. Let $f : \mathbb{I} \times \mathfrak{I} \to \mathfrak{R}^n$ be a continuous function defined by $(t, \mathbf{x}) \mapsto f(t, \mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

Definition 2.1 (**System of ODEs**) Assume Hypothesis(H_s) on $\mathbf{f}$. A first order system of n ordinary differential equations is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t). \tag{2.1}$$

A dynamical system can be divided into autonomous system and non-autonomous system.

Definition 2.2 (**Equilibrium Point**) The point $\bar{\mathbf{x}} \in \mathbb{R}^n$ is an equilibrium point for the differential equation $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$ if $\mathbf{f}(\bar{\mathbf{x}}, t) = 0$ for all t.

2.3 Phase Plane

In applied mathematics, in particular the context of nonlinear system analysis, a phase plane is a visual display of certain characteristics of certain kinds of differential equations, a coordinate plane with axes being the values of the two state variables, say (x, y) or (p, q) etc.(any pair of variables). It is a two-dimensional case of the general $n-$ dimensional phase space.

The phase plane method refers to graphically determining the existence of limit cycles in the solutions of the differential equation.

The solutions to the differential equation are a family of functions. Graphically, this can be plotted in the phase plane a two dimensional vector field. Vectors representing the derivatives of the points with respect to parameters(say time t), that is $(\frac{dx}{dt}, \frac{dy}{dt})$ at representative points are drawn. With enough to these arrows in place the system behavior over the regions of plane in analysis can be visualized and limit cycles can be easily identified. So the phase planes are useful in visualizing the behavior of physical systems.

2.4 Phase Portrait, Orbit and Attractor

Definition 2.3 (**Phase Portrait:**) A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point.

Phase portraits are an invaluable tool in studying the system of differential equations. They consist of a plot of typical trajectories in the state space. This reveals information that the said dynamical system is stable, uniformly stable, asymptotically stable or unstable.

Definition 2.4 (**Orbit:**) In mathematics, in the study of dynamical systems, an orbit is a collection of points related by the evolution function of the dynamical system.

Definition 2.5 (**Attractor:**) In the mathematical field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if slightly disturbed.

2.5 Steady States

A constant solution $y(t) = c$ is called a steady state for a differential equation. It is a solution where the value of y does not change over time. For example, consider the differential equation $\dot{y} = ay$. In order for the level of y to be the same this year and last year, we must have that y does not change, or $\dot{y} = 0$. Therefore, the only value of y for which this can happen (as long as $a \neq 0$) is $y = 0$, and so $y = 0$ is a steady state to the equation.

For the system of differential equations

$$\dot{x} = f_1(x, y),\ \dot{y} = f_2(x, y) \tag{2.2}$$

, we can find the steady state (equilibrium solution) of the system by setting both $\dot{x} = 0$ and $\dot{y} = 0$. A critical point (x^*, y^*) of the dynamical system (2.2) is a point such that $f_1(x^*, y^*) = f_2(x^*, y^*) = 0$. If (x^*, y^*) is a critical point of the system, then the constant-value functions $x(t) = x^*$, $y(t) = y^*$ satisfy the equations in (2.2). Such a constant-valued solution is called an equilibrium solution of the dynamical system (2.2). The critical point also called equilibrium point of (2.2).

Example 2.1 Find the steady state point (Or Equilibrium Point) for the system of equations $\dot{x} = e^{(x-1)} - 1$ and $\dot{y} = ye^x$.

Solution: Setting both these equations equal to 0 yields

$$\begin{aligned} \dot{x} &= 0 \Rightarrow e^{(x-1)} = 1 \Rightarrow x = 1, \\ \dot{y} &= 0 \Rightarrow ye^x = 0 \Rightarrow y = 0. \end{aligned}$$

Therefore $(1, 0)$ is the steady state point (Or Equilibrium Point) of the given system.

Example 2.2 Find the steady state point (Or Equilibrium Point) for the system of equations $\dot{x} = x + 2y$ and $\dot{y} = x^2 + y$.

Solution: Setting both these equations equal to 0 yields

$$\dot{x} = 0 \Rightarrow x = 2y,$$
$$\dot{y} = 0 \Rightarrow y = -x^2.$$
$$\text{So,} \qquad x = -2(-x^2) \Rightarrow x(1-2x) = 0 \Rightarrow x = \{0, \frac{1}{2}\}$$
$$\text{and} \qquad y = \{0, -\frac{1}{4}\}$$

Therefore $(0,0), (\frac{1}{2}, -\frac{1}{4})$ are steady state points (Or Equilibrium Point) of the given system.

2.6 Differential Dynamical Structure of multi variable form for autonomous system

Consider the general system of differential equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \dot{x}_i(t) &= f_i(x_1, x_2, \cdots, x_n), \ (i = 1, 2, \cdots, n) \end{aligned} \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ depends on t and $\mathbf{f} = (f_1, f_2, \cdots, f_n)$.

Definition 2.6 (**Lyapunov stability:**) Let $\mathbf{x}^*(t)$ be a given real or complex solution vector (or trajectories) of the above $n-$ dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then

1. $\mathbf{x}^*(t)$ is a Lyapunov stable for if only if to each value of $\epsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ϵ) such that

$$||\mathbf{x}(t_0) - \mathbf{x}^*(t_0)|| < \delta \ \Rightarrow ||\mathbf{x}(t) - \mathbf{x}^*(t)|| < \epsilon \quad , (\forall t > t_0) \tag{2.4}$$

   where $\mathbf{x}(t)$ represents any other neighbouring solution (or trajectories).
2. Otherwise the solution (or trajectories) $\mathbf{x}^*(t)$ is unstable in the sense of Lyapunov.

Note. The stable solutions (or trajectories) of an autonomous system are also uniformly stable, since the system is invariant with respect to time translation.

Definition 2.7 (**Quasi-asymptotic stability:**) Let $\mathbf{x}^*$ be a solution (or trajectories) for $t \geq t_0$. If additionally there exists $\delta > 0$ such that

$$||\mathbf{x}(t_0) - \mathbf{x}^*(t_0)|| < \delta \ \Rightarrow \lim_{t\to\infty} ||\mathbf{x}(t) - \mathbf{x}^*(t)|| = 0 \tag{2.5}$$

then the solution (or trajectories) is said to be quasi-asymptotic stable.

Definition 2.8 (**Asymptotic stability:**) Let $\mathbf{x}^*$ be a stable solution (or trajectories) for $t \geq t_0$. If additionally there exists $\delta > 0$ such that

$$||\mathbf{x}(t_0) - \mathbf{x}^*(t_0)|| < \delta \ \Rightarrow \lim_{t\to\infty} ||\mathbf{x}(t) - \mathbf{x}^*(t)|| = 0 \tag{2.6}$$

then the solution (or trajectories) is said to be asymptotic stable.

Note. The solution (trajectories) is asymptotically stable when it is both stable and quasi-asymptotically stable.

Example 2.3 Show that all solutions of the n-dimensional system $\dot{x} = -x$ are stable for $t \geq 0$ in the Lyapunov sense.

Solution: The general solution is given by

$$x(t) = x(0)e^{-t}$$

Consider the stability of $x^*(t)$ for $t > 0$ where

$$x^*(t) = x^*(0)e^{-t}$$

For any $x^*(0)$, we have

$$\|x(t) - x^*(t)\| \leq \|x(0) - x^*(0)\|e^{-t} \leq \|x(0) - x^*(0)\|, \ t \geq 0.$$

Therefore, given any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\|x(0) - x^*(0)\| < \delta \Rightarrow \|x(t) - x^*(t)\| < \epsilon, \ t > 0.$$

So the solutions of the system are stable in Lyapunov sence. Here, $\epsilon = \delta > 0$. The solutions are also uniformly stable.

Example 2.4 Show that all solutions of the n-dimensional system $\dot{x} = -x$ are asymptotically stable for $t \geq 0$ in the Lyapunov sense.

Solution: The general solution is given by

$$x(t) = x(0)e^{-t}$$

Consider the stability of $x^*(t)$ for $t > 0$ where

$$x^*(t) = x^*(0)e^{-t}$$

For any $x^*(0)$, we have

$$\|x(t) - x^*(t)\| \leq \|x(0) - x^*(0)\|e^{-t} \leq \|x(0) - x^*(0)\|, \ t \geq 0.$$

Therefore, given any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\|x(0) - x^*(0)\| < \delta \Rightarrow \|x(t) - x^*(t)\| < \epsilon, \ t > 0.$$

Here, $\epsilon = \delta > 0$. Hence, the solutions of the system are stable and

$$\|x(0) - x^*(0)\| < \delta \Rightarrow \lim_{t \to \infty} \|x(t) - x^*(t)\| = 0.$$

Hence the stable solutions of the system are asymptotically stable.

2.7 Stability Analysis of Single Variable for Autonomous System

We call a steady state y^* in the domain asymptotically stable if $\exists\, r > 0$ and $B(y^*, r) \subset$ domain such that if we have as an initial point any $y \in B(y^*, r)$, the system will converge to y^* over time. We call a system stable if all points in the domain converge to a steady state.

It is easy to tell the stability of a steady state from the phase diagram: If the arrows point towards the steady state from both sides, it is stable. Else it is unstable. A simple test for stability is as follows:
Let $\dot{y} = f(y)$.
If $\frac{df(y)}{dy}\,|_{y^*} < 0$, then the steady state y^* is stable.
If $\frac{df(y)}{dy}\,|_{y^*} > 0$, then the steady state y^* is unstable.

Example 2.5 Determine the steady state and their stability of the differential equation $\dot{y} = f(y) = y^2 - 5y + 6$.

Solution: Let $\dot{y} = 0$. Then $0 = y^2 - 5y + 6 = (y-2)(y-3)$. Therefore, we have two steady states, $y = 2$ and $y = 3$. Next, take the derivative of $\dot{y} = y^2 - 5y + 6$ with respect to y to get

$$\frac{df(y)}{dy} = 2y - 5. \tag{2.7}$$

Evaluated at $y^* = 2$, we have

$$\frac{df(y)}{dy}\,|_{y^*} = 2(2) - 5 = -1 < 0. \tag{2.8}$$

Therefore $y^* = 2$ is a stable steady state. Evaluated at $y^* = 3$, we have

$$\frac{df(y)}{dy}\,|_{y^*} = 2(3) - 5 = 1 > 0 \tag{2.9}$$

and we have that $y^* = 3$ is an unstable steady state.

2.8 Stability Analysis for Autonomous System (General Method)

Equilibria are not always stable. Since stable and unstable equilibria play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability. Suppose that we have a set of autonomous ordinary differential equations, written in vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.10}$$

Equilibrium point: A point $\mathbf{x}^*$ is called an equilibrium point if $\frac{d\mathbf{x}}{dt} = 0$ at $\mathbf{x}^*$ i.e., if $\mathbf{f}(\mathbf{x}^*) = 0$.
Suppose that $\mathbf{x}^*$ is an equilibrium point. By the definition of equilibrium points $\mathbf{f}(\mathbf{x}^*) = 0$. Now

suppose that we take a multivariate Taylor expansion of the right-hand side of our differential equation:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}^*) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\mathbf{x}^*}(\mathbf{x}-\mathbf{x}^*) + \cdots \\ &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\mathbf{x}^*}(\mathbf{x}-\mathbf{x}^*) + \cdots \end{aligned} \tag{2.11}$$

The partial derivative in the above equation is to be interpreted as the Jacobian matrix. If the components of the state vector $\mathbf{x}$ are $(x_1, x_2, x_3, \cdots, x_n)$ and the components of the rate vector $\mathbf{f}$ are $(f_1, f_2, \cdots, f_n)$, then the jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & . & . & . & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & . & . & . & \frac{\partial f_2}{\partial x_n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & . & . & . & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{2.12}$$

Now we define $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}^*$. Taking a derivative of this definition we have $\dot{\delta\mathbf{x}} = \dot{\mathbf{x}}$. If $\delta\mathbf{x}$ is small, then only the first term in equation (2.11) is significant since the higher terms involve powers of our small displacement from equilibrium. If we want to know how trajectories behave near the equilibrium point, *e.g.* whether they move toward or away from the equilibrium point, it should therefore be good enough to keep just this term. Then we have

$$\dot{\delta\mathbf{x}} = J^*\delta\mathbf{x} \tag{2.13}$$

where J^* is the Jacobian evaluated at the equilibrium point. The matrix J^* is a constant, so this is just a linear differential equation. According to the theory of linear differential equations, the solution can be written as a superposition of terms of the form $e^{\lambda_j t}$ where λ_j is the set of eigenvalues of the Jacobian.
The eigenvalues of the Jacobian are, in general, complex numbers. Let $\lambda_j = \mu_j + i\nu_j$, where μ_j and ν_j are respectively, the real and imaginary parts of the eigenvalue. Each of the exponential terms in the expansion can therefore be written

$$e^{\lambda_j t} = e^{\mu_j t} e^{i\nu_j t} \tag{2.14}$$

The complex exponential can be written as

$$e^{i\nu_j t} = cos(\nu_j t) + isin(\nu_j t). \tag{2.15}$$

Definition 2.9 (**Hyperbolic Equilibrium point:**) An equilibrium point $\mathbf{x}^*$ is called a hyperbolic equilibrium point of (2.10) if none of the eigenvalues of the jacobian matrix J^* have zero real part.

Definition 2.10 (**Non Hyperbolic Equilibrium point:**) An equilibrium point $\mathbf{x}^*$ is called a non hyperbolic equilibrium point of (2.10) if one of the eigenvalues of the jacobian matrix J^* have zero real part.

Definition 2.11 (**Node :**) An equilibrium point $\mathbf{x}^*$ is called a node when all eigenvalues of the corresponding jacobian matrix are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive).

Definition 2.12 (**Focus :**) An equilibrium point $\mathbf{x}^*$ is called a Focus (sometimes called spiral point) when eigenvalues of the corresponding jacobian matrix are complex-conjugate; The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.

Definition 2.13 (**Sink:**) An equilibrium point $\mathbf{x}^*$ of (2.10) is called a sink if all of the eigenvalues of the matrix J^* have negative real part.

Definition 2.14 (**Source:**) An equilibrium point $\mathbf{x}^*$ of (2.10) is called a source if all of the eigenvalues of the matrix J^* have positive real part.

Definition 2.15 (**Saddle point:**) An equilibrium point $\mathbf{x}^*$ of (2.10) is called a saddle if at least one of the eigenvalues of the matrix J^* have positive real part and at least one of the eigenvalues of the eigenvalues of the matrix J^* have negative real part.

The complex part of the eigenvalue therefore only contributes an oscillatory component to the solution. So the nature of the solution depends completely on the value of μ_j. Depending the value of μ_j there may exist three cases.

Case 1: If $\mu_j < 0$ for all j, then all solution of the system (2.10) will tend to the equilibrium point and the equilibrium point is called **stable equilibrium point**.

Case 2: If $\mu_j > 0$ for any j, $e^{\mu_j t}$ grows with time, which means that trajectories will tend to move away from the equilibrium point and the equilibrium point is called unstable equilibrium point.

Definition 2.16 Unstable Equilibrium: An equilibrium point $\mathbf{x}^*$ is called unstable equilibrium if it is not a stable equilibrium.

These two cases leads us to a very important theorem which is given below:

Theorem 2.1 An equilibrium point $\mathbf{x}^*$ of the differential equation (2.10) is stable if all the eigenvalues of J^*, the Jacobian evaluated at $\mathbf{x}^*$, have negative real parts. The equilibrium point is unstable if at least one of the eigenvalues has a positive real part.

Case 3: If $\mu_j = 0$ for some j, then the general theorem is silent on the issue of what happens if some of the eigenvalues have zero real parts (non hyperbolic equilibrium point) while the others are all negative.

2.8.1 Stability of Linear Differential Dynamical System with Constant Coefficient

To check the stability of a linear dynamical system, we remember the following theorem 2.2.

Theorem 2.2 (Jordan and Smith (2009)) Let A be a constant matrix in the system $\dot{\mathbf{x}} = A\mathbf{x}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.
(i) If the system is stable, then $Re(\lambda_i) \leq 0$, $i = 1, 2, \cdots n$.
(ii) If either $Re(\lambda_i) < 0$, $i = 1, 2, \cdots n$ or if $Re(\lambda_i) \leq 0$, $i = 1, 2, \cdots n$ and there is no zero repeated eigenvalues, then the system is uniformly stable.
(iii) The system is asymptotically stable if and only if $Re(\lambda_i) < 0$, $i = 1, 2, \cdots n$ (and then it is also uniformly stable by (ii)).
(iv) If $Re(\lambda_i) > 0$ for any i, the solution is unstable.

Note. The uniformly stable is discussed in details in section 2.9.1.

Example 2.6 Discuss about the stability of the following system of differential equations:

$$\dot{x} = -x + y, \qquad \dot{y} = 4x - y \tag{2.16}$$

Solution: For this problem , (0, 0) is an equilibrium point and the above system can be rewritten as

$$\dot{X} = AX \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation for A is

$$(1 + \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0$$

and therefore the eigenvalues of the matrix A are $\lambda = (-3, 1)$. Therefore, we know that the system is unstable and the equilibrium point (0, 0) is also called saddle point.

Example 2.7 Find the eigenvalues for the system and show that the system is stable

$$\dot{y_1} = -y_1 - 2y_2, \qquad \dot{y_2} = 2y_1 - y_2 VU(CBCS)(2018)$$

Solution: For this problem , (0, 0) is an equilibrium point and the variational matrix at the origin is given by

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

and the eigenvalues are $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$. Both eigenvalues are complex with negative real part, so the system is stable and the equilibrium point (0, 0) is also called sink.

2.8.2 Stability of Non-linear Differential Dynamical System

To check the stability of a non-linear dynamical system, we have remember the following jacobian.

Let J be the jacobian matrix of the system of differential equations (2.3) which is defined by

$$J = \frac{df}{dy} = \left[\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}.$$

Then we can test for stability as follows:
y^* is stable iff all eigenvalues of $J(y^*)$ are negative or have negative real parts.
y^* is unstable iff some eigenvalues of $J(y^*)$ is positive or has positive real parts.

If the Jacobian at y^* has some pure imaginary or zero eigenvalues and no positive eigenvalues, then we cannot determine the stability of the steady state for looking at Jacobian.

Example 2.8 Examine the stability of the system of differential equations $\dot{x} = e^{x-1} - 1$ and $\dot{y} = ye^x$.

Solution: We already calculated the steady state of the system will be $z = (x, y) = (1, 0)$. The Jacobian of the given system is

$$\begin{pmatrix} e^{x-1} & 0 \\ ye^x & e^x \end{pmatrix} \tag{2.17}$$

When $z = (1, 0)$, then we have the Jacobian

$$\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$$

which implies that the eigenvalues of the system are 1 and e. Since both of these are positive, so the equilibrium point is called source and the system is an unstable at (1, 0).(shown in figure-2.1)

Example 2.9 Examine the stability of the system of differential equations $\dot{x} = x + 2y$ and $\dot{y} = x^2 + y$.

Solution: We already calculated that the steady states of the system are $z = (x, y) = \{(0, 0), (\frac{1}{2}, -\frac{1}{4})\}$. The Jacobian of the system is

$$\begin{pmatrix} 1 & 2 \\ 2x & 1 \end{pmatrix}$$

When $z = (0, 0)$, then we have the Jacobian

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

which implies that the repeated eigenvalues is 1. Since both of these are positive, so the equilibrium point $(0, 0)$ is called a source and the system is an unstable at this point.
When $z = (\frac{1}{2}, -\frac{1}{4})$, then we have the Jacobian

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

and solving for eigenvalues yields $\lambda = 1 \pm \sqrt{2}$. Since one of them is positive, so the equilibrium point $(\frac{1}{2}, -\frac{1}{4})$ is called a saddle point and the system is an unstable at this point (shown in figure-2.2).

Example 2.10 Examine the stability of the system of differential equations $\dot{x} = e^{1-x} - 1$ and $\dot{y} = (2 - y)e^x$.

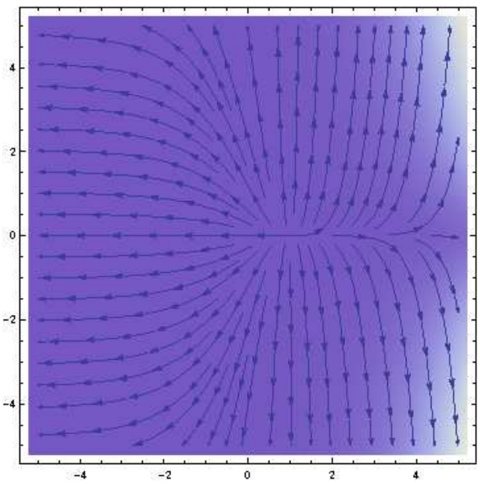

Figure 2.1: Unstable steady state

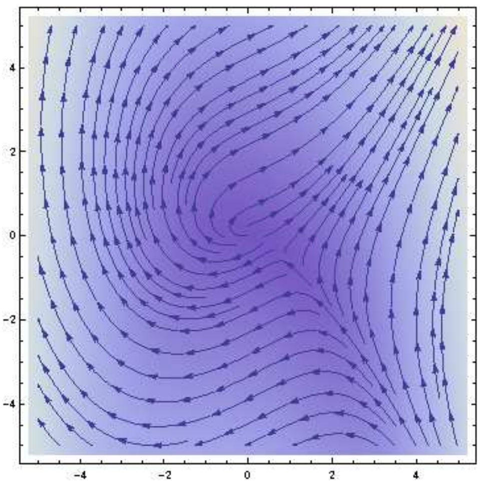

Figure 2.2: Unstable steady state

Solution: We already calculated that the steady state of the system will be $z = (x, y) = (1, 2)$. The Jacobian of the system is

$$\begin{pmatrix} -e^{1-x} & 0 \\ (2-y)e^{x} & -e^{x} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -e \end{pmatrix}$$

Which implies that the eigenvalues of the system are -1 and $-e$. Since both of these are negative, so the equilibrium point $(1, 2)$ is called a sink and the system is stable at this point (shown in figure- 2.3).

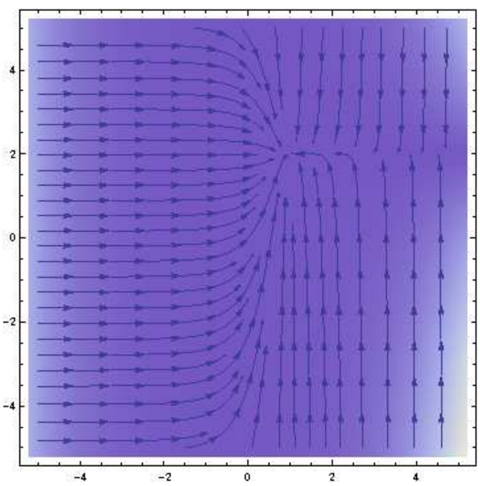

Figure 2.3: Stable steady state

Theorem 2.3 (**Lyapunov's Stability**) Consider an autonomous system described by the vector differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.18}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x}))^T$. Suppose the origin of the state space is an equilibrium point. Consider a *Lyapunov* function V that is characterized the following properties:

1. $V(\mathbf{0}) = 0$
2. $V(\mathbf{x}) > 0$ in a neighborhood $\Re^n$ of the origin.
3. V is continuous in $\Re^n$ and possesses continuous first partial derivatives $\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}$.

The time derivative of V following the trajectories of the vector differential equation is given by

$$\frac{dV(\mathbf{x})}{dt} = \dot{V}(\mathbf{x}) = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \cdots + \frac{\partial V}{\partial x_n} f_n \tag{2.19}$$

$$\dot{V}(\mathbf{x}) = \left[\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right]^T \dot{\mathbf{x}} \tag{2.20}$$

where $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$ is the gradient vector of the *Lyapunov* function V given by

$$\left[\frac{\partial V(\mathbf{x})}{\partial x}\right]^T = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \text{ and } \dot{x}^T = (f_1(x), f_2(x), \cdots, f_n(x))$$

with a suitable chosen function it may be possible to show that $\dot{V}(\mathbf{x}) < \mathbf{0}\,(\dot{V} \leq \mathbf{0})$ for $\mathbf{x} \neq \mathbf{0}$. Since $\mathbf{x} = \mathbf{0}$ is an equilibrium point i.e., $f_1(0) = f_2(0) = \cdots = f_n(0) = 0$ and so $\dot{V}(\mathbf{0}) = \mathbf{0}$. In this case the origin is uniformly and asymptotically stable equilibrium point(only uniformly stable) where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ depends on t.

Proof. The elegant proof has been given by Jordan and Smith (2009). The reader may skip the proof of the above theorem for his first reading.

Example 2.11 Investigate the stability of the zero solution of the system

$$\dot{x} = -y - x^3, \;\; \dot{y} = x - y^3 \tag{2.21}$$

Solution: Since $\dot{x} = 0$ and $\dot{y} = 0$ at (0, 0), so origin is an equilibrium point.
Let us consider a family of curves

$$V(x, y) = x^2 + y^2 = \alpha, \;\; 0 < \alpha < \infty$$

$$\text{Then } \dot{V}(x, y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 2x(-y - x^3) + 2y(x - y^3) = -2(x^4 + y^4).$$

which is negative everywhere except at the origin. Therefore we have found a strong Lyapunov function for the system. Hence the zero solution is the uniformly and asymptotically stable in the Lyapunov sense. Also the stream density plot of phase portrait is presented in Fig.-2.4 which also show the stability of the said dynamical system.

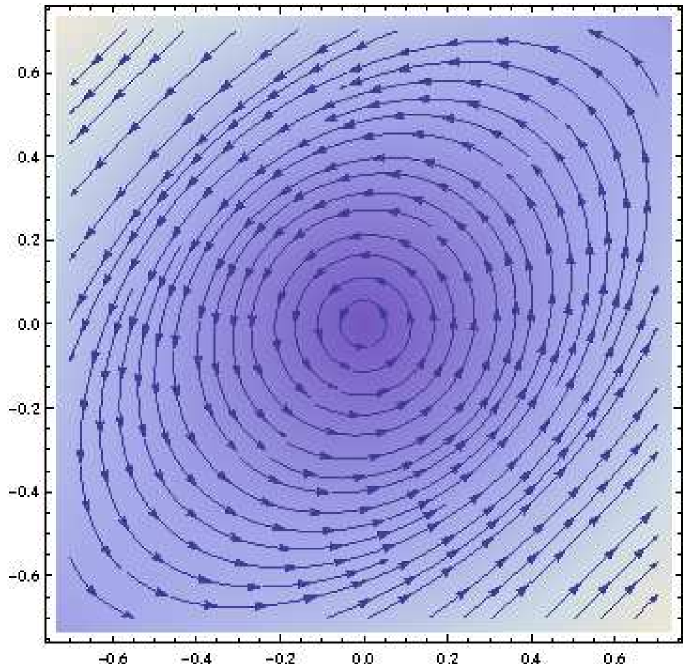

Figure 2.4: The stream density plot of phase portrait of the differential equation (2.21)

Example 2.12 Consider the system

$$\dot{x_1} = -2x_2 + x_2x_3 - x_1^3, \qquad \dot{x_2} = x_1 - x_1x_3 - x_2^3, \qquad \dot{x_3} = x_1x_2 - x_3^3.$$

Solution: The given problem can be written as

$$\begin{aligned} f_1(x_1, x_2, x_3) &= -2x_2 + x_2x_3 - x_1^3 \\ f_2(x_1, x_2, x_3) &= x_1 - x_1x_3 - x_2^3 \\ f_3(x_1, x_2, x_3) &= x_1x_2 - x_3^3 \end{aligned}$$

For this problem , origin $(0, 0, 0)$ is an equilibrium point and the variational matrix at the origin is given by

$$J(\mathbf{0}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.22}$$

The eigenvalues of the variational matrix are $\lambda_1 = 0, \lambda_{2,3} = \pm 2i$ i.e., $\mathbf{x}^* = (0, 0, 0)$ is a non-hyperbolic equilibrium point. So, we use Lyapunov function method and a liapnunov function is constructed as follows

$$V(\mathbf{x}) = x_1^2 + 2x_2^2 + x_3^2$$

satisfies $V(\mathbf{x}) > 0$ and

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_3}\dot{x}_3 \\ &= 2x_1(-2x_2 + x_2x_3 - x_1^3) + 4x_2(x_1 - x_1x_3 - x_2^3) + 2x_3(x_1x_2 - x_3^3) \\ &= -2(x_1^4 + 2x_2^4 + x_3^4) < 0\end{aligned}$$

for $\mathbf{x} \neq \mathbf{0}$. Therefore according to the theorem 2.3, the system is asymptotically stable at origin.

Theorem 2.4 (Jordan and Smith (2009)) If the n-dimensional system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x})$ with A, a constant $n \times n$ matrix and

$$\begin{aligned}&(i)\text{ the zero solution of } \dot{\mathbf{x}} = A\mathbf{x} \text{ is an asymptotically stable;} \\ &(ii)\ \mathbf{h}(\mathbf{0}) = 0 \text{ and } \lim_{\|\mathbf{x}\|\to 0} \frac{\|\mathbf{h}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0\end{aligned} \tag{2.23}$$

then $\mathbf{x}(t) = 0$, $t \geq t_0$ for any t_0 is an asymptotically stable solution of

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x}) \tag{2.24}$$

Proof. We have to show there is a neighborhood of the origin where $\mathbf{V}(\mathbf{x})$ defined by $\mathbf{V}(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x}$, and $\mathbf{K} = \int_0^\infty \mathbf{e}^{\mathbf{A^Tt}}\mathbf{e}^{\mathbf{At}}dt$ is a strong Lyapunov function for (2.24). The function $\mathbf{V}$ given by

$$\mathbf{V}(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x},$$

where $\mathbf{K} = \int_0^\infty \mathbf{e}^{\mathbf{A^Tt}}\mathbf{e}^{\mathbf{At}}dt$ is positive definite when (i) holds. Also for (2.24),

$$\begin{aligned}\mathbf{V}(x) &= \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{K}\dot{\mathbf{x}} \\ &= \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{K} + \mathbf{K}\mathbf{A})\mathbf{x} + \mathbf{h}^{\mathbf{T}}\mathbf{K}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{h} = -\mathbf{x}^{\mathbf{T}}\mathbf{x} + \mathbf{2h}^{\mathbf{T}}(\mathbf{x})\mathbf{K}\mathbf{x},\end{aligned} \tag{2.25}$$

by $\mathbf{A}^{\mathbf{T}}\mathbf{K} + \mathbf{K}\mathbf{A} = -\mathbf{I}$, and the symmetry of $\mathbf{K}$. We have to display a neighbourhood of the origin in which the first term of (2.25) dominates. From the Cauchy-Schwarz inequality we have

$$|\mathbf{2h}^{\mathbf{T}}(\mathbf{x})\mathbf{K}\mathbf{x}| \leq 2\|\mathbf{h}(\mathbf{x})\|\|\mathbf{K}\|\|\mathbf{x}\|. \tag{2.26}$$

By (ii), given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|\mathbf{x}\| < \delta \ \Rightarrow \|\mathbf{h}(\mathbf{x})\| < \epsilon\|\mathbf{x}\|,$$

so that from (2.26),

$$|\mathbf{2h}^{\mathbf{T}}(\mathbf{x})\mathbf{K}\mathbf{x}| \leq \mathbf{2}\epsilon\|\mathbf{K}\|\|\mathbf{x}\|^2.. \tag{2.27}$$

Let ϵ to be chosen so that

$$\epsilon < \frac{1}{(4\|\mathbf{K}\|)},$$

then from (2.27),

$$\|\mathbf{x}\| < \delta \Rightarrow |\mathbf{2h}^{\mathbf{T}}(\mathbf{x})\mathbf{K}\mathbf{x}| < \frac{\|\mathbf{x}\|}{2} = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$$

Therefore, by (2.25). $\dot{V}(x)$ for (2.24) is negative definition on $||\mathbf{x}|| < \delta$. By Theorem 2.3, the zero solution is asymptotically stable.

Example 2.13 Prove that the zero solution of the system

$$\dot{x}_1 = -x_1 + x_2^2 + x_3^2,\ \dot{x}_2 = x_1 - 2x_2 + x_1^2,\ \dot{x}_3 = x_1 + 2x_2 - 3x_3 + x_2x_3$$

is uniformly and asymptotically stable.

Solution: For this problem , $(0,0,0)$ is an equilibrium point and the variational matrix at the origin is given by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 2 & -3 \end{pmatrix} \tag{2.28}$$

and $\mathbf{h}(\mathbf{x}) = \left[x_2^2 + x_3^2, x_1^2, x_2x_3\right]^T$.
The eigenvalues of A are $-1, -2, -3$. Therefore the zero of $\dot{x} = Ax$ is uniformly and asymptotically stable. Also $||\mathbf{h}(\mathbf{x})|| = x_2^2 + x_3^2 + x_1^2 + |x_2x_3|$ and $\mathbf{h}(0,0,0) = 0$, $\lim\limits_{||\mathbf{x}||\to 0} \frac{||\mathbf{h}(\mathbf{x})||}{||\mathbf{x}||} = \lim\limits_{||\mathbf{x}||\to 0} \frac{x_2^2+x_3^2+x_1^2+|x_2x_3|}{(|x_1|+|x_2|+|x_3|)} = 0$. Then by Theorem-2.4, the zero solution of the given system is uniformly and asymptotically stable.

Theorem 2.5 If the n-dimensional system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x})$ with A is a constant $n \times n$ matrix and

1. the eigenvalues of A are distinct, none are zero, and at least one has positive real part and
2. $\lim\limits_{||\mathbf{x}||\to 0} \frac{||\mathbf{h}(\mathbf{x})||}{||\mathbf{x}||} = 0$,

then the zero solution $\mathbf{x}(t) = 0$, $t \geq t_0$ of the regular system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x}). \tag{2.29}$$

is unstable.

Proof. The elegant proof has been given by Jordan and Smith (2009). The reader may skip the proof of the above theorem for his first reading.

Example 2.14 Prove that the zero solution of the system

$$\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = \frac{d^2x}{dt^2}(x + \frac{dx}{dt})$$

is unstable.

Solution: Write as the equivalent system

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = z,\ \frac{dz}{dt} = -2x + y + 2z + z(x + y)$$

Then $(0,0,0)$ is the equilibrium point for the above problem and corresponding variational matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \tag{2.30}$$

and $h(x, y, z) = z(x + y)$.
The eigenvalues of A are $-1, 1, 2$, two of which are positive. Also $\|\mathbf{h}(x, y, z)\| = |z(x + y)|$ and $\mathbf{h}(0, 0, 0) = 0$, $\lim\limits_{\|(x,y,z)\|\to 0} \frac{\|\mathbf{h}(x,y,z)\|}{\|(x,y,z)\|} = \lim\limits_{\|(x,y,z)\|\to 0} \frac{|z(x+y)|}{(|x|+|y|+|z|)} = 0$. Therefore, by Theorem-2.5, the zero solution of the given system is unstable.

2.9 Stability Analysis of non-autonomous system

The stability of a non-autonomous dynamical system have been discussed in the following ways.

2.9.1 Stability Analysis of non-autonomous system for multi variables

Definition 2.17 (**Lyapunov stability:**) Let $\mathbf{x}^*(t)$ be a given real or complex solution vector (or trajectories) of the $n-$ dimensional system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$. Then

1. $\mathbf{x}^*(t)$ is a Lyapunov stable for $t \geq t_0$ if and only if to each value of $\epsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ϵ and t_0) such that
$$\|\mathbf{x}(t_0) - \mathbf{x}^*(t_0)\| < \delta \;\Rightarrow \|\mathbf{x}(t) - \mathbf{x}^*(t)\| < \epsilon \quad , (\forall t > t_0) \tag{2.31}$$
where $\mathbf{x}(t)$ represents any other neighbouring solution (or trajectories).
2. If the given system is autonomous, the reference to t_0 in 1 may be disregarded, the solution (or trajectories) $\mathbf{x}^*(t)$ is either Lyapunov stable, or not for all t_0.
3. Otherwise the solution (or trajectories) $\mathbf{x}^*(t)$ is unstable in the sense of Lyapunov.

Definition 2.18 (**Uniform stability:**) If a solution (or trajectories) is stable for $t \geq t_0$ and the δ of Definition 2.17 is independent of t_0, the solution (or trajectories) is uniformly stable on $t \geq t_0$.

Note. It is clear that any stable solutions (or trajectories) of an autonomous system are uniformly stable, since the system is invariant with respect to time translation.

Definition 2.19 (**Quasi-asymptotic stability:**) Let $\mathbf{x}^*$ be a solution (or trajectories) for $t \geq t_0$. If additionally there exists $\delta(t_0) > 0$ such that
$$\|\mathbf{x}(t_0) - \mathbf{x}^*(t_0)\| < \delta \;\Rightarrow \lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| = 0 \tag{2.32}$$
then the solution (or trajectories) is said to be quasi-asymptotic stable.

Definition 2.20 (**Asymptotic stability:**) Let $\mathbf{x}^*$ be a stable solution (or trajectories) for $t \geq t_0$. If additionally there exists $\delta(t_0) > 0$ such that
$$\|\mathbf{x}(t_0) - \mathbf{x}^*(t_0)\| < \delta \;\Rightarrow \lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| = 0 \tag{2.33}$$
then the solution (or trajectories) is said to be asymptotic stable.

Note. The solution (trajectories) is asymptotically stable when it is both stable and quasi-asymptotically stable.

The origin is called *stable* when for every $\epsilon > 0$ and $\delta(\epsilon, t_0) > 0$ exists so that for all $\|\mathbf{x}(t_0)\| < \delta$ it follows that $\|\mathbf{x}(t)\| < \epsilon$ for all $t \geq t_0$. If δ is only the function of ϵ and is independent to t_0, then the origin is uniformly stable.

A system for which the right hand sides of the differential equation system are not explicit functions of time but functions of $\mathbf{x}$ only, i.e.

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}) \tag{2.34}$$

is 'autonomous'. If an equilibrium point of an autonomous system is stable it is also uniformly stable.

The origin is *quasi-asymptotically stable* when a quantity $\delta(t_0) > 0$ exists so that $\|\mathbf{x}(t_0)\| < \delta$ implies

$$\lim_{t\to\infty} x(t) = 0 \tag{2.35}$$

The origin is *asymptotically stable* when it is both *stable* and *quasi-asymptotically stable*.

When every solution (or trajectories) from every point for which solutions (or trajectories) can be defined tends to the origin, the origin is said to be 'asymptotically stable in the large'. When this applies for all points in the stable space the origin is said to be 'asymptotically stable in the whole'. This distinction in Russian literature has been lost in the translation into English of both these properties as 'global asymptotic stability'.

Example 2.15 Show that all solutions of the n-dimensional system $\dot{x} = -tx$ are stable for $t \geq t_0$ in the Lyapunov sense.

Solution: The general solution is given by

$$x(t) = x(t_0)e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})}$$

Consider the stability of $x^*(t)$ for $t > t_0$ where

$$x^*(t) = x^*(t_0)e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})}$$

For any $x^*(t_0)$, we have

$$\|x(t) - x^*(t)\| \leq \|x(t_0) - x^*(t_0)\|e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})} \leq \|x(t_0) - x^*(t_0)\|, \ t \geq t_0.$$

Therefore, given any $\epsilon > 0$,

$$\|x(t_0) - x^*(t_0)\| < \epsilon \Rightarrow \|x(t) - x^*(t)\| < \eta, \ t > t_0.$$

So the solutions of the system are stable in *Lyapunov* sense. Here, $\epsilon = \eta > 0$. Hence, the solutions are also uniformly stable.

Example 2.16 Show that all solutions of the n-dimensional system $\dot{x} = -tx$ are asymptotically stable for $t \geq t_0$ in the Lyapunov sense.

Solution: The general solution is given by

$$x(t) = x(t_0)e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})}$$

Consider the stability of $x^*(t)$ for $t > t_0$ where

$$x^*(t) = x^*(t_0)e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})}$$

For any $x^*(t_0)$, we have

$$\|x(t) - x^*(t)\| \leq \|x(t_0) - x^*(t_0)\|e^{-(\frac{t^2}{2} - \frac{t_0^2}{2})} \leq \|x(t_0) - x^*(t_0)\|, \ t \geq t_0.$$

Therefore, given any $\epsilon > 0$,

$$\|x(t_0) - x^*(t_0)\| < \epsilon \Rightarrow \|x(t) - x^*(t)\| < \eta, \ t > t_0.$$

Here, $\epsilon = \eta > 0$. Hence, the solutions of the system are uniformly stable and

$$\|x(t_0) - x^*(t_0)\| < \epsilon \Rightarrow \lim_{t\to\infty} \|x(t) - x^*(t)\| = 0.$$

Hence the stable solutions of the system are asymptotically stable.

Theorem 2.6 (Jordan and Smith (2009)) If the n-dimensional system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x}, t)$ with A, a constant $n \times n$ matrix and

1. the zero solution(hence every solution by Theorem-2.7) of $\dot{\mathbf{x}} = A\mathbf{x}$ is an asymptotically stable;
2. $\mathbf{h}(0, t) = \mathbf{0}$ and $\lim_{\|\mathbf{x}\|\to 0} \frac{\|\mathbf{h}(\mathbf{x},t)\|}{\|\mathbf{x}\|} = 0$,

then $\mathbf{x}(t) = \mathbf{0}$ for $t \geq 0$ is an asymptotically stable solution of the regular system $\dot{\mathbf{x}} = A\mathbf{x}+\mathbf{h}(\mathbf{x}, t)$.

Example 2.17 Prove that when $a > 0$ and $b > 0$, all solutions of

$$\ddot{x} + a\dot{x} + bx + cx^2e^{-t} \cos t = 0$$

are asymptotically stable for $t \geq t_0$, for any t_0.

Solution: Let us consider $y = \dot{x}$. Then the appropriate equivalent system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -cx^2e^{-t} \cos\, t \end{pmatrix}. \tag{2.36}$$

Comparting (2.36) with $\dot{X} = AX + \mathbf{h}(X, t)$, we get

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \ and \ \mathbf{h}(X, t) = \begin{pmatrix} 0 \\ -cx^2e^{-t} \cos\, t \end{pmatrix}.$$

The eigenvalues of A are $\frac{-a\pm \sqrt{a^2-4b}}{2a}$. Given that $a > 0$ and $b > 0$, so real parts of eigenvalues are negative. Also $(0, 0)$ is the equilibrium point of (2.36). Therefore, the zero of $\dot{X} = AX$ is uniformly and asymptotically stable. Also $\|\mathbf{h}(X, t)\| = |cx^2e^{-t} \cos\, t|$ and $\mathbf{h}(0, t) = 0$, $\lim_{\|X\|\to 0} \frac{\|\mathbf{h}(X,t)\|}{\|X\|} = \lim_{(x,y)\to(0,0)} \frac{|cx^2e^{-t} \cos\, t|}{(|x|+|y|)} = 0$. Then by Theorem-2.6, the zero solution of the given non-homogeneous system is uniformly and asymptotically stable.

2.9.2 Stability relation between homogeneous and nonhomogeneous systems

Consider the n-dimensional nonhomogeneous time-varying linear differential equations:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{f}(t) \tag{2.37}$$

and the corresponding homogeneous time-varying linear differential equations:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \tag{2.38}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, $A(t) = (a_{ij}(t))_{n\times n} \in C[I, \Re^{n\times n}]$, $f(t) = (f_1(t), \cdots, f_n(t))^T \in C[I, \Re^n]$.

Definition 2.21 If all solutions of (2.37) have same stability, then systems (2.37) is said to stable with this class of stability.

Theorem 2.7 (Jordan and Smith (2009)) All solutions of the regular linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{2.39}$$

have the same Lyapunov stability property (unstable, stable, uniformly stable, asymptotically stable, uniformly and asymptotically stable). This is the same as that of the zero(or any other) solution of the homogeneous equation $\dot{\xi} = \mathbf{A}(t)\xi(t)$ provided that $\mathbf{f}$ is bounded.

Proof. First, we consider to investigate the stability of a solution $\mathbf{x}^*(t)$. Let $\mathbf{x}(t)$ represent any other solution, and define $\xi(t)$ by

$$\dot{\xi}(t) = \mathbf{x}(t) - \mathbf{x}^*(t). \tag{2.40}$$

Then $\xi(t)$ tracks the difference between the 'test' solution and a solution having a different initial value at time t_0. The initial condition for ξ is

$$\dot{\xi}(t_0) = \mathbf{x}(t_0) - \mathbf{x}^*(t_0). \tag{2.41}$$

Also, ξ satisfies the homogeneous equation derived from (2.39):

$$\dot{\xi}(t) = \mathbf{A}(t)\xi. \tag{2.42}$$

By comparison of 2.40, 2.41 and 2.42 with Definition 2.17, it can be seen that the Lyapunov stability property of $\mathbf{x}^*(t)$ is the same as the stability of the zero solution of 2.42. $\xi(t)$ is called a perturbation of the solution $\mathbf{x}^*(t)$.
Since this new formulation of the problem is independent of the solution of 2.39 initially chosen, hence the Theorem 2.7.

Note.: The Theorem 2.7 is also true for unbounded function $\mathbf{f}$.

Theorem 2.8 (Jordan and Smith (2009)) For the regular linear system $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, the zero solution (and hence, by Theorem-2.7, all solutions) are Liopunov stable on $t \geq t_0$, t_0 arbitrary, if and only if every solution is bounded as $t \to \infty$. If A is constant and every solution is bounded, the solutions are uniformly stable.

Proof. First, suppose that the zero solution, $x^*(t) = 0$ is stable. Choose any $\epsilon > 0$. Then there exists a corresponding δ for Definition 2.17. Let

$$\Psi(t) = [\psi_1(t), \psi_2(t), \cdots, \psi_n(t)]$$

be the fundamental matrix satisfying the initial condition

$$\Psi(t_0) = \frac{\delta \mathbf{I}}{2}$$

where **I** is the unit matrix. (This is a diagonal matrix with element $\frac{\delta}{2}$ on the principal diagonal.) By Definition 2.17

$$\Psi_i(t_0) = \frac{1}{2}\delta < \delta \Rightarrow \|\Psi_i(t)\| < \epsilon, t \geq t_0.$$

Therefore every solution (By Theorem 2.7) is bounded since any other solution is a linear combination of the $\Psi_i(t)$.

Conversely, suppose that every solution is bounded. Let $\Phi(t)$ be any fundamental matrix, then there exists, by hypothesis, $M > 0$ such that $\|\Phi(t) < M, t \geq t_0\|$. Given any $\epsilon > 0$ let

$$\delta = \frac{\epsilon}{M\|\Phi^{-1}(t_0)\|}.$$

Let $\mathbf{x}(t)$ be any solution, we will test the stability of the zero solution. By Theorem **??** in Chapter **??**, We have $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}(t_0)$, and if $\|\mathbf{x}(t)\| < \delta$, then

$$\|\mathbf{x}(t)\| \leq \|\Phi(t)\|\|\Phi^{-1}(t_0)\|\|\mathbf{x}(t_0)\| < M\frac{\delta}{M\delta}\delta = \epsilon.$$

Thus Definition 2.17 of stability for the zero solution is satisfied.

When **A** is a constant matrix, the autonomous nature of the system ensures that stability is uniform.

Example 2.18 Explain why all solutions are uniformly stable but not asymptotically stable.

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x + \cos t.$$

Solution: The homogeneous system of differential equation is

$$\dot{\xi} = \eta, \qquad \dot{\eta} = -\omega^2 \xi \tag{2.43}$$

The corresponding matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \tag{2.44}$$

and the eigenvalues of A are $\pm i\omega$. By calculating the corresponding eigenvectors, the general solution is

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ i\omega \end{pmatrix} e^{i\omega t} + c_2 \begin{pmatrix} 1 \\ -i\omega \end{pmatrix} e^{-i\omega t} \tag{2.45}$$

Hence the real parts of the eigenvalues of the corresponding homogeneous equations are zero, therefore the solutions of homogeneous system of equations are uniformly stable but not asymptotically stable. As $\cos t$ is bounded, so by Theorem-2.8, all solutions of given non-homogeneous equation are also uniformly stable but not asymptotically stable.

Example 2.19 Explain why all solutions are uniformly and asymptotically stable.

$$\frac{dY_1}{dt} = -Y_1 + 5Y_2 + e^{-t}, \qquad \frac{dY_2}{dt} = -4Y_1 - 5Y_2$$

Solution: The corresponding homogeneous equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} -1 & 5 \\ -4 & -5 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} -1-\lambda & 5 \\ -4 & -5-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 25 = 0$$

so that $\lambda = -3 \pm 4i$. Next, we need the eigenvector for $\lambda = -3 + 4i$:

$$\begin{pmatrix} 2-4i & 5 \\ -4 & -2-4i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (2-4i)v_1 + 5v_2 \\ -4v_1 - (2+4i)v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the above equations, we get

$$\mathbf{v} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Hence, $e^{\lambda t}v = e^{-3t}(\cos 4t + i\sin 4t)\left\{\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 4 \end{pmatrix}\right\}$.
Therefore, the complementary solution is

$$\begin{aligned} y(t) &= A\,Re(e^{\lambda t}v) + B\,Im(e^{\lambda t}v) \\ &= Ae^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\sin 4t\right) + Be^{-3t}\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}\cos 4t\right) \end{aligned}$$

Therefore,

$$\begin{aligned} y_1(t) &= 5e^{-3t}(A\cos 4t + B\sin 4t) \\ y_2(t) &= 2e^{-3t}\Big[(-A+2B)\cos 4t - (2A+B)\sin 4t\Big] \end{aligned}$$

Hence the real parts of the eigenvalues of the corresponding homogeneous system of equations are negative, so the zero solution corresponding to the equilibrium point $\dot{y}_1 = \dot{y}_2 = 0$ is uniformly and asymptotically stable i.e., all solutions of the corresponding homogeneous equations are bounded and e^{-t} is alway bounded for all $t \in R$, so by Theorem-2.8, all solutions of given non-homogeneous equations are uniformly and asymptotically stable. Also the stream plot of phase portrait is presented in Fig.-2.5 which shown the stability of the said dynamical system.

Theorem 2.9 (Jordan and Smith (2009)) Suppose that
(i) $\mathbf{A}$ be a constant $n \times n$ matrix whose eigenvalues have negative real parts,
(ii) For $t_0 \le t < \infty$, $\mathbf{C}(t)$, is continuous and

$$\int_{t_0}^{t} \|\mathbf{C}(t)\| dt \text{ is bounded.} \tag{2.46}$$

Then all solutions of linear, homogeneous system $\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{C}(t))\mathbf{x}(t)$ are asymptotically stable.

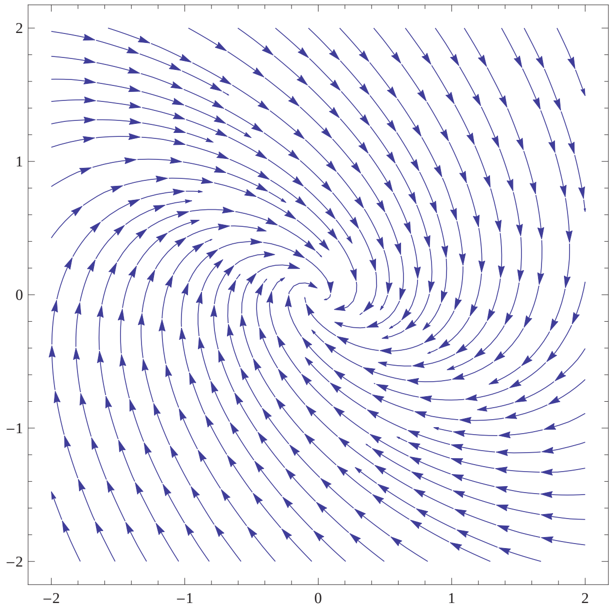

Figure 2.5: The stream plot of phase portrait of the differential equation (2.46)

Proof. Write the system in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{C}(\mathbf{t})\mathbf{x} \tag{2.47}$$

If $\mathbf{x}(t)$ is a solution, then $\mathbf{C}(\mathbf{t})\mathbf{x}(\mathbf{t})$ is a function of t which may play the part of $\mathbf{f}(t)$ in Theorem **??**. Therefore (2.47) implies

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t - s + t_0)\Phi^{-1}(t_0)\mathbf{C}(s)\mathbf{x}(s)ds,$$

where Ψ is any fundamental matrix for the system $\mathbf{x} = \mathbf{A}\mathbf{x}$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. Using the properties of norms which parallel the properties of vector magnitudes, we obtain

$$||\mathbf{x}(t)|| \le ||\Phi(t)||||\Phi^{-1}(t_0)||||\mathbf{x}_0|| + ||\Phi^{-1}(t_0)|| \int_{t_0}^{t} ||\Phi(t - s + t_0)||||\mathbf{C}(s)||||\mathbf{x}(s)||ds \tag{2.48}$$

Since $\mathbf{A}$ has eigenvalues with negative real parts, Theorem 2.2 shows that for some positive M and m,

$$||\Phi(t)|| \le Me^{-mt}, \quad t \ge t_0. \tag{2.49}$$

Therefore, putting $||\Phi^{-1}(t_0)|| = \beta$, in equation (2.48) implies, after some regrouping, that for $t \ge t_0$

$$||\mathbf{x}(t)||e^{mt} \le M\beta||\mathbf{x}_0|| + \int_{t_0}^{t} \{||\mathbf{x}(s)e^{ms}||\}\{||\mathbf{C}(s)||M\beta e^{mt_0}\}ds \tag{2.50}$$

Then from (2.50) and Theorem **??**,

$$\begin{aligned} ||\mathbf{x}(t)||e^{mt} &\leq M\beta||\mathbf{x}_0|| \exp\Big(\beta M e^{-mt_0} \int_{t_0}^{t} ||\mathbf{C}(s)||ds\Big) \\ \text{or } ||\mathbf{x}(t)|| &\leq M\beta||\mathbf{x}_0|| \exp\Big(\beta M e^{-mt_0} \int_{t_0}^{t} ||\mathbf{C}(s)||ds - mt\Big). \end{aligned} \tag{2.51}$$

Therefore, by (2.46) every solution of (2.47) is bounded for $t \geq t_0$ and is therefore stable by Theorem 2.8. Also every solution tends to zero as $t \to \infty$ and is therefore asymptotically stable.

Example 2.20 Show that when $a > 0$ and $b > 0$ all solutions of

$$\ddot{x} + a\dot{x} + (b + ce^{-t}\cos t)x = 0$$

are asymptotically stable for $t \geq t_0$ for any t_0.

Solution: The appropriate equivalent system (with $\dot{x} = y$) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -ce^{-t}\cos t & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

In the notation of the above Theorem 2.9,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \ \mathbf{C}(t) = \begin{pmatrix} 0 & 0 \\ -ce^{-t}\cos t & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$ which are negatives if $a > 0$ and $b > 0$. Also

$$\int_{t_0}^{\infty} ||\mathbf{C}(t)||dt = |c| \int_{t_0}^{\infty} e^{-t}|\cos t|dt < \infty.$$

The conditions of the Theorem 2.9 are satisfied, so all solutions are asymptotically stable.

Corollary 2.1 (Jordan and Smith (2009)) If $\mathbf{C}(t)$ satisfies the conditions of the Theorem 2.9 but all solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ are merely bounded, then all solutions of $\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{C}(t))\mathbf{x}(t)$ are bounded and therefore stable.
Proof. This follows from (2.51) by writing $m = 0$ in (2.50). Note that $Re\{\lambda_i\} \leq 0$ for all i is not in itself sufficient to establish the boundedness of all solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. □

Example 2.21 Show that all solution $\ddot{x} + \{a + c(1 + t^2)^{-1}\}x = e^{-t}$ are stable if $a > 0$.

Solution: The appropriate equivalent system (with $\dot{x} = y$) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -c(1+t^2)^{-1} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

By Theorem 2.7, all solutions of the given system have the same stability property as the zero solution (or any other) of the corresponding homogeneous system $\dot{\xi} = \{\mathbf{A} + \mathbf{C}(t)\}\xi$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}, \ \mathbf{C}(t) = \begin{pmatrix} 0 & 0 \\ -c(1+t^2)^{-1} & 0 \end{pmatrix}.$$

The solutions of $\dot{\xi} = \mathbf{A}\xi$ are bounded when $a > 0$ (the zero solution is a centre on the phase plane). Also

$$\int_{t_0}^{\infty} \|\mathbf{C}(t)\| dt = |c| \int_{t_0}^{\infty} \frac{dt}{1+t^2} < \infty.$$

By Corollary 2.1, all solutions of $\dot{\xi} = \{\mathbf{A} + \mathbf{C}(t)\}\xi$ are bounded and are therefore stable.

2.9.3 Lyapunov's method for determining stability of the zero solution for Non Autonomous Systems

Consider the continuous time dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{2.52}$$

with $\mathbf{f}(\mathbf{0}, t) = \mathbf{0}$ for all t (which means that the origin is an equilibrium point). Assume that there exists a scalar function $\mathbf{V}$ with continuous partial derivatives in $x_1, x_2, \cdots, x_n$ and t and such that $\mathbf{V}(\mathbf{0}, t) = \mathbf{0}$ for all t. Furthermore, the following conditions should be satisfied.

1. $\mathbf{V}(\mathbf{x}, t)$ should be positive definite. This means that there exists a continuous, non-decreasing scalar function $\alpha(\|\mathbf{x}\|)$ such that $\alpha(0) = 0$ for all t and $0 < \alpha(\|\mathbf{x}\|) < \mathbf{V}(\mathbf{x}, t)$ for all $x \neq 0$.

2. There exists a continuous non-decreasing function $\beta(\|\mathbf{x}\|)$ such that $\beta(0) = 0$ for all t and $\mathbf{V}(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$ for all $\mathbf{x} \neq \mathbf{0}$.

3. There exists a scalar function $\gamma(\|\mathbf{x}\|)$ such that $\gamma(0) = 0$ and $\gamma(\|\mathbf{x}\|) > 0$ for $\mathbf{x} \neq \mathbf{0}$, and along each trajectory of equation (2.52) the following inequality is satisfied:

$$\frac{d\mathbf{V}(\mathbf{x}, t)}{dt} = \left[\frac{\partial \mathbf{V}(\mathbf{x}, t)}{\partial \mathbf{x}}\right] \times \mathbf{f}(\mathbf{x}, t) + \frac{\partial \mathbf{V}(\mathbf{x}, t)}{\partial t} < -\gamma(\|\mathbf{x}\|) < 0 \ \mathbf{x} \neq \mathbf{0} \tag{2.53}$$

4. $\alpha(\|\mathbf{x}\|)$ and $\beta(\|\mathbf{x}\|)$ tend to infinity as $\|\mathbf{x}\| \to \infty$.
   If these conditions are fulfilled, then the equilibrium point $\mathbf{x} = \mathbf{0}$ is *uniformly asymptotically stable in the large* and $\mathbf{V}(\mathbf{x}, t)$ is called a Lyapunov function of the system described by equation (2.52).

Example 2.22 Consider the asymptotic stability of the solution of the system

$$\frac{dx}{dt} = -\frac{x}{t + \sin x} \ \text{ in } [t_0, \ \infty] \tag{2.54}$$

Solution: Choose $V(t,x) = (t + \sin x)x^2$, $\alpha(\|x\|) = x^2|\sin x|$ and $\beta(\|x\|) = x^2$, $(t_0 \geq 2)$. Then $V(t,0) = 0$, $\alpha(0) = 0$, $\beta(0) = 0$, $\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|)$ and

$$\begin{aligned} \frac{dv}{dt} &= 2(t+\sin x)x\left(\frac{-x}{t+\sin x}\right) + x^2\left(1+\cos x \times \frac{-x}{t+\sin x}\right) \\ &= -x^2\left(1+\frac{x\cos x}{t+\sin x}\right) < -\gamma(\|x\|) < 0, \ (\|x\| << 1) \end{aligned}$$

where $\gamma(\|x\|) = x^2\left(1+\frac{x\cos x}{t+\sin x}\right)$ with $\gamma(\|0\|) = 0$ and since $t_0 \geq 2$. Also $\alpha(\|x\|)$ and $\beta(\|x\|)$ tend to infinity as $\|x\| \to \infty$. So the equilibrium point $x = 0$ is asymptotically stable and $V(x,t) = (t+\sin x)x^2$ is called a Lyapunov function of the system described by equation (2.54).

2.9.4 Stability of Non-autonomous system: Liopunov stability

Consider a regular dynamical system, not necessarily autonomous in n dimensions, written in vector form as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) \tag{2.55}$$

Suppose that $\bar{\mathbf{x}}(t)$ and $\mathbf{x}(t)$ be two real or complex solution vector of (2.55) with components

$$\begin{aligned} \bar{\mathbf{x}}(t) &= \left[\bar{x}_1(t), \bar{x}_2(t), \cdots, \bar{x}_n(t)\right]^T \\ \mathbf{x}(t) &= \left[x_1(t), x_2(t), \cdots, x_n(t)\right]^T. \end{aligned}$$

(where T stands for transpose: these are column vectors). The separation between them at any time t will be denoted by the symbol $\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\|$, denoted by

$$\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| = \left(\sum_{i=1}^{n} |x_i(t) - \bar{x}_i(t)|^2\right)^{\frac{1}{2}}, \tag{2.56}$$

where $|\cdots|$ denotes the modulus in the complex number sense, and the ordinary magnitude when x_i and $\bar{x}_i$ are real solutions.

2.10 Periodic Solution of a Dynamical System

Periodic Orbit: A periodic orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time. A dynamical system exhibiting a stable periodic orbit is often called an oscillator.

Definition 2.22 Limit Cycle: In mathematics, in the study of dynamical systems with two-dimensional phase space, a limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. Such behavior is exhibited in some nonlinear systems. Limit cycles have been used to model the behavior of a great many real world oscillatory systems.

Definition 2.23 Stable Limit Cycle: In the case where all the neighboring trajectories approach the limit cycle as time approaches infinity, it is called a stable or attractive limit cycle. Stable limit cycles are examples of attractors. They imply self-sustained oscillations: the closed trajectory describes perfect periodic behavior of the system, and any small perturbation from this closed trajectory causes the system to return to it, making the system stick to the limit cycle.

Example 2.23 Consider the system $\dot{x} = y + x(1 - x^2 - y^2), \qquad \dot{y} = x + y(1 - x^2 - y^2)$.

Solution: Then the above system can be expressed as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x(x^2 + y^2) \\ -y(x^2 + y^2) \end{pmatrix}$$

The corresponding linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

has eigenvalues $1 \pm i$ and has an unstable focus at $(0, 0)$. It is useful in this case to transform to polar coordinates (r, θ), $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Differentiating these two expression with respect to t and simplifying we have the expression

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = -1 \tag{2.57}$$

From the polar differential equations, $\dot{r} = 0$ when $r = 0$ or $r = 1$, $r = 0$ corresponds to the critical point $(0, 0)$ but $r = 1$ corresponds to a periodic cycle $x^2 + y^2 = 1$, a circle. From the equation of $\dot{r}, \dot{\theta}$ we see that $\dot{r} > 0$ for $0 < r < 1$, $\dot{r} < 0$ for $r > 1$ and $\dot{\theta} < 0$ for all r, θ. Therefore r is increasing with t inside the unit circle, and decreasing with t outside.
All trajectories converge to the periodic trajectory $r = 1$ as $t \to \infty$, $r = 1$ is an asymptotically stable limit cycle (Fig. 2.6) of the system.

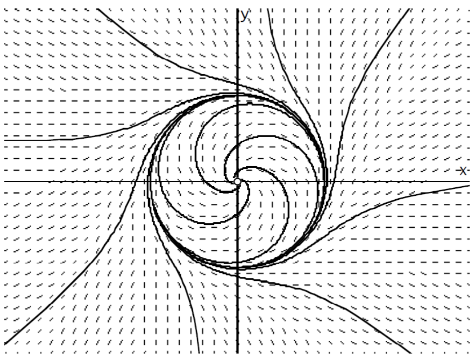

Figure 2.6: The stable limit cycle.

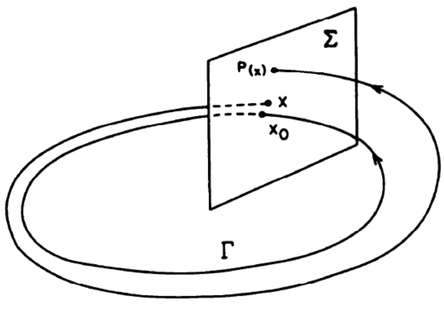

Figure 2.7: Poincare Map.

Definition 2.24 Unstable Limit Cycle: If instead all neighboring trajectories approach it as time approaches negative infinity, then it is an unstable limit cycle.

Definition 2.25 Semi stable Limit Cycle: If there is a neighboring trajectory which spirals into the limit cycle as time approaches infinity, and another one which spirals into it as time approaches negative infinity, then it is a semi-stable limit cycle.

Definition 2.26 Heteroclinic cycle: In mathematics, a heteroclinic cycle is an invariant set in the phase space of a dynamical system. It is a topological circle of equilibrium points and connecting heteroclinic orbits. If a heteroclinic cycle is asymptotically stable, approaching trajectories spend longer and longer periods of time in a neighbourhood of successive equilibria.

Definition 2.27 Poincar'e Map or orbital Map: Probably the most basic tool for studying the stability and bifurcations of periodic orbits is the Poincare map which is shown in Fig. 2.7 .

Example 2.24 Consider the system $\dot{x} = -y + x(1 - x^2 - y^2), \qquad \dot{y} = x + y(1 - x^2 - y^2)$.

Solution: It has a limit cycle Γ represented by $\gamma(t) = (\cos t, \sin t)^T$. The poincar'e map can be found by solving this system written in polar coordinate

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

with $r(0) = r_0$ and $\theta(0) = \theta_0$. Then solving the above two equations, we get

$$\begin{aligned} r(t, r_0) &= \left[1 + (\frac{1}{r_0^2} - 1)e^{-2t}\right]^{-1/2}, \\ \theta(t, \theta_0) &= t + \theta_0 \end{aligned}$$

If Σ is the ray $\theta = \theta_0$ through the origin, then Σ is perpendicular to Γ and the trajectory through the point $(r_o, \theta_0) \in \Sigma \bigcap \Gamma$ at $t = 0$ intersects the ray $\theta = \theta_0$ again at $t = 2\pi$. It follows that the Poincar'e map is given by

$$P(r_0) = \left[1 + (\frac{1}{r_0^2} - 1)e^{-4\pi}\right]^{-1/2}$$

clearly $P(1) = 1$ corresponding to the cycle γ and it is shown in Fig. 2.8. We see that

$$P'(r_0) = e^{-4\pi} r_0^{-3} \left[1 + (\frac{1}{r_0^2} - 1)e^{-4\pi}\right]^{-3/2}$$

and that $P'(1) = e^{-4\pi} < 1$.

2.11 Linearization

Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point. We have well established tools to analyze and stabilize linear systems.

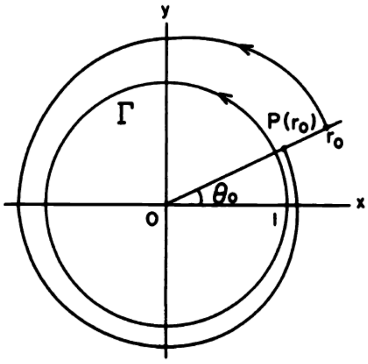

Figure 2.8: Poincar'e Map for the above example.

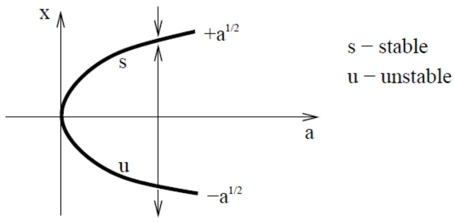

Figure 2.9: Saddle node bifurcation for the above example.

2.11.1 Method

Let us write the general form of nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ as:

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, \cdots, x_n)\\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \cdots, x_n)\\ \vdots \quad & \quad \vdots \quad \vdots\\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \cdots, x_n) \qquad (2.58)\end{aligned}$$

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has equilibrium state $\mathbf{x}_e = [x_{1e}, x_{2e}, \cdots, x_{ne}]^T$ such that $\mathbf{f}(\mathbf{x}_e) = 0$.
We now perturb the equilibrium state by $\mathbf{x} = \mathbf{x}_e + \xi$ where $\xi = [\xi_1, \xi_2, \cdots, \xi_n]^T$. Taylor's series expansion yields

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{f}(\mathbf{x}_e + \xi)\\ &= \mathbf{f}(\mathbf{x}_e) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_e} \xi + O(\|\xi\|^2) \qquad (2.59)\end{aligned}$$

where

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_e} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{array}\right]_{\mathbf{x}_e}$$

are the Jacobian of $\mathbf{f}$ and $O(\|\xi\|^2)$ is an order of $\|\xi\|$ such that $\lim_{\|\xi\|\to 0} \frac{O(\|\xi\|^2)}{\|\xi\|} = 0$. Here,

$$\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}_e}{dt} + \frac{d\xi}{dt} = \frac{d\xi}{dt} \qquad (2.60)$$

because $\mathbf{x}_e$ is constant. Further $\mathbf{f}(\mathbf{x}_e) = 0$.
Let

$$A = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_e}.$$

Neglecting higher order terms, we arrive at the linear approximation of (2.59) by (2.60)

$$\frac{d\xi}{dt} = A\xi. \tag{2.61}$$

Note: All solutions of the nonlinear autonomous system (2.58) have the same Lyapunov stability property (unstable, stable, uniformly stable, asymptotically stable, uniformly and asymptotically stable). This is the same as that of the zero(or any other) solution of the homogeneous equation $\frac{d\xi}{dt} = A\xi$.

Example 2.25 By linearizing around the critical points, draw the phase plane portrait of

$$y'' + y - y^3 = 0 \tag{2.62}$$

Solution: First, rewrite in first order form so let $y_1 = y$ and define $y_2 = y_1'$, now, from the equation $y_1'' = y_2' = y_1^3 - y_1$, putting these together gives:

$$y_1' = y_2, \qquad y_2' = y_1^3 - y_1 \tag{2.63}$$

The stationary points occur when $y_1' = y_2' = 0$, hence $y_2 = 0$ and $y_1^3 - y_1 = 0$, or, when $y_1 = -1$ or $y_1 = 0$ or $y_1 = 1$. We will look at each of these stationary points in turn. Near $y_1 = -1$ and $y_2 = 0$ we have $y_1 = -1 + \eta$ where η is small. Hence

$$y_2' = (-1+\eta)^3 - (-1+\eta) \approx -1 + 3\eta + 1 - \eta = 2\eta \tag{2.64}$$

and so, near this stationary point, the system is approximately

$$\begin{array}{rcl} \eta' & = & y_2 \\ y_2' & \approx & 2\eta \end{array} \tag{2.65}$$

$$\text{or, } \begin{pmatrix} \eta' \\ y_2' \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \tag{2.66}$$

The matrix here has eigenvalues $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$ with eigenvectors

$$\mathbf{y}_{11} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \tag{2.67}$$

$$\text{and} \qquad \mathbf{y}_{21} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \tag{2.68}$$

Hence, this stationary point is a saddle point and provided η remains small, it is approximated by

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} \approx C_1\mathbf{y}_{11}e^{\sqrt{2}t} + C_2\mathbf{y}_{21}e^{-\sqrt{2}t} \tag{2.69}$$

Near $y_1 = 0$ and $y_2 = 0$ we assume both y_1 and y_2 are small and make the approximation

$$y_2' = y_1^3 - y_1 \approx -y_1 \tag{2.70}$$

and so, near this stationary point, the system is approximately

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.71}$$

This matrix has eigenvalues $\pm i$ and so this is a circle node (Stable node). Near $y_1 = 1$ and $y_2 = 0$ we have $y_1 = 1 + \eta$ where η is small. Hence

$$y_2' = (1+\eta)^3 - (1+\eta) \approx 2\eta \tag{2.72}$$

and so, near this stationary point, the system is approximately

$$\begin{pmatrix} \eta' \\ y_2' \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ y_2 \end{pmatrix} \tag{2.73}$$

and so this point is the same as the $y_1 = -1$, $y_2 = 0$ stationary point. Also the phase diagram is shown in Fig.-2.10

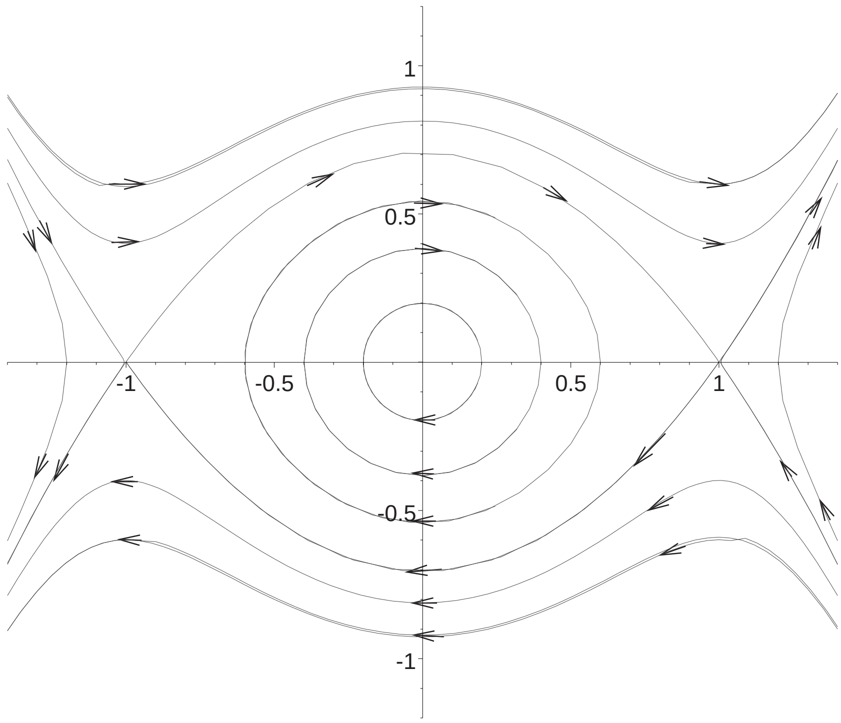

Figure 2.10: The phase portrait of the given differential equation

Example 2.26 By linearizing around the critical points, draw the phase plane portrait and check the stability of

$$y'' = \cos 2y \tag{2.74}$$

Solution: First order system:

$$y_1' = y_2, \qquad y_2' = \cos 2y_1 \tag{2.75}$$

now, the critical points are located where $y_1' = y_2' = 0$. This happens when $y_2 = 0$ and $\cos 2y_1 = 0$, that means $2y_1 = (2n+1)\pi/2$ where n is an odd integer, or $y_1 = (2n+1)\pi/4$ where again n is an odd integer.

Near $y_1 = \pi/4$ write $y_1 = \pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(\pi/4 + \eta) = -\sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system becomes

$$\eta' = y_2, \qquad y_2' = -2\eta.$$

The corresponding matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \tag{2.76}$$

and the eigenvalues of A are $\pm i\sqrt{2}$. By calculating the corresponding eigenvectors, the general solution is

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ i\sqrt{2} \end{pmatrix} e^{i\sqrt{2}t} + c_2 \begin{pmatrix} 1 \\ -i\sqrt{2} \end{pmatrix} e^{-i\sqrt{2}t} \tag{2.77}$$

and by beginning at $\eta = r$ and $y_2 = 0$ we get

$$\begin{pmatrix} \eta \\ y_2 \end{pmatrix} = r \begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t \end{pmatrix} \tag{2.78}$$

Then by Theorem-2.8, we have the given system is uniformly stable at the equilibrium point $(\frac{\pi}{4}, 0)$ and the said point is also an ellipse with the vertical $\sqrt{2}$ times as long as the horizontal.
Near $y_1 = 3\pi/4$ write $y_1 = 3\pi/4 + \eta$ and use $\cos 2y_1 = \cos 2(3\pi/4 + \eta) = \sin 2\eta$ and this linearizes as $\sin 2\eta \sim 2\eta$ so the system becomes

$$\eta' = y_2, \qquad y_2' = 2\eta \tag{2.79}$$

This is a saddle-point with eigenvalues $\pm\sqrt{2}$ and corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix}. \tag{2.80}$$

Then by Theorem-2.8, we have the given system is unstable at the equilibrium point $(\frac{3\pi}{4}, 0)$. This pattern repeats by periodicity, the phase portrait is shown in Fig.-2.11

2.12 Bifurcation Theory

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior.

Co-dimension of a bifurcation: The co-dimension of a bifurcation is the number of parameters which must be varied for the bifurcation to occur. This corresponds to the co-dimension of the parameter set for which the bifurcation occurs within the full space of parameters. Saddle-node bifurcations and Hopf bifurcations are the only generic local bifurcations which are really co-dimension-one (the others all having higher codimension). However, transcritical and pitchfork bifurcations are also often thought of as codimension-one, because the normal forms can be written with only one parameter.

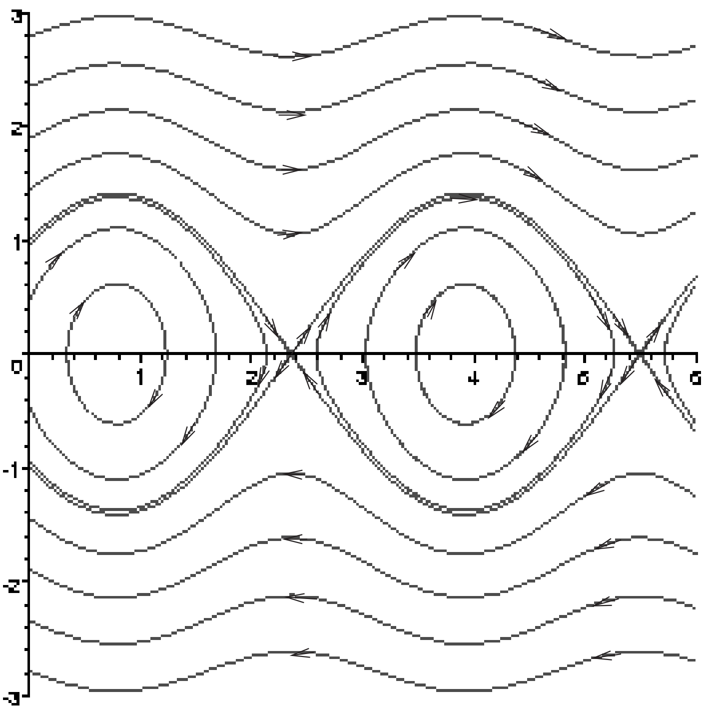

Figure 2.11: The phase portrait of the differential equation (2.74)

2.12.1 Classification of Bifurcation

In this section we examine further parametric bifurcations of first-order systems, mainly in two variables. The approach is through examples, and no attempt is made to cover general bifurcation theory. However, we can first make some general observations about the first-order autonomous system in n real variables which contains m real parameters. This can be expressed in the form

$$\dot{\mathbf{x}} = X(\mu, \mathbf{x}), \ \mathbf{x} \in \Re^n, \ \mu \in \Re^n \tag{2.81}$$

where μ is an m- dimensional vector of real parameters (the letter $\Re$ stands for the set of real numbers). Equilibrium points occur at the solutions for $\mathbf{x}$ of the n scalar equations expressed in vector form by

$$X(\mu, \mathbf{x}) = 0 \tag{2.82}$$

for any given μ.
Suppose that $(\mu_0, \mathbf{x}_0)$ is a solution of this equation. Then $\mu = \mu_0$ is a bifurcation point if the structure of the phase diagram changes as μ passes through μ_0. This rather imprecise definition covers a number of possibilities including a change in the number of equilibrium points as μ passes through μ_0, or a change in their stability.

There are different types of bifurcation. It is mainly divided into two types such as
(*i*) Local bifurcations
(*ii*) Global bifurcations.

(i) Local Bifurcation:

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighborhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local').

Local bifurcation is divided into many parts such as (*a*) **Saddle-node (fold) bifurcation**, (*b*) **Transcritical Bifurcation**, (*c*) **Period-doubling (flip) bifurcation**, (*d*)**Hopf bifurcation**.

(*a*) **Saddle node bifurcation:**
In the mathematical area of bifurcation theory a saddle-node bifurcation, tangential bifurcation or fold bifurcation is a local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other. The term 'saddle-node bifurcation' is most often used in reference to continuous dynamical systems. In discrete dynamical systems, the same bifurcation is often instead called a fold bifurcation. Another name is blue skies bifurcation in reference to the sudden creation of two fixed points.

Example 2.27 Consider the dynamical system defined by

$$\frac{dx}{dt} = r + x^2, \quad \text{where } r, x \text{ are reals.} \tag{2.83}$$

Solution: Here $f(x) = r + x^2$, $f'(x) = 2x$ where x is the state variable and r is the bifurcation parameter.

(i)If $r < 0$ there are two equilibrium points, a stable equilibrium point at $-\sqrt{-r}$ and an unstable one at $+\sqrt{-r}$.
(ii) At $r = 0$ (the bifurcation point) there is exactly one equilibrium point. At this point the fixed point is no longer hyperbolic. In this case the fixed point is called a saddle-node fixed point (See in Figure 2.12).
(iii) If $r > 0$ there are no equilibrium points.
In fact, this is a normal form of a saddle-node bifurcation. A scalar differential equation $\frac{dx}{dt} = f(r, x)$ which has a fixed point at $x = 0$ for $r = 0$ with $\frac{\partial f}{\partial x}(0,0) = 0$ is locally topological equivalent to $\frac{dx}{dt} = r \pm x^2$, provided it satisfies $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$ and $\frac{\partial f}{\partial r}(0,0) \neq 0$. The first condition is the non degeneracy condition and the second condition is the transversality condition.

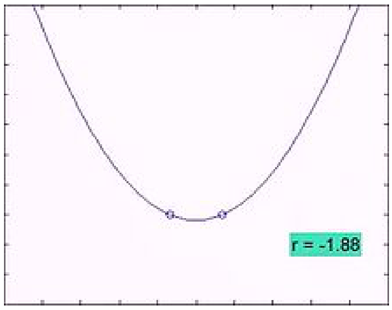

Figure 2.12: A normal saddle node bifurcation

Example 2.28 Consider the equations

$$\dot{x} = y, \ \dot{y} = x^2 - y - \mu, \ \text{where} \ \mu \text{ is real.} \tag{2.84}$$

Solution: Equilibrium points occur where $y = 0, x^2 - y - \mu = 0$. Geometrically they lie on the intersection of the plane $y = 0$ and the surface $y = x^2 - \mu$ in (x, y, μ) space. In this case, all the equilibrium points lie in the plane $y = 0$, so that we need only show the curve $x^2 = \mu$ of equilibrium points in the (x, μ) plane as shown in Fig. 2.13 . There is a bifurcation point at $\mu = \mu_0 = 0$ where $x = x_0 = 0, y = y_0 = 0$. For $\mu < 0$ there are no equilibrium points whilst for $\mu > 0$ there are two at $x = \pm\sqrt{\mu}$.
For $\mu > 0$, let $x = \pm\sqrt{\mu} + x^{'}$, $y = y^{'}$ in (2.84). Hence, to the first order

$$\dot{x}^{'} = y^{'}, \ \dot{y}^{'} = (\pm\sqrt{\mu} + x^{'})^2 - y^{'} - \mu \approx \pm 2x^{'}\sqrt{\mu} - y^{'}. \tag{2.85}$$

The linear approximation is given by

$$\begin{bmatrix} \dot{x}' \\ \dot{y}' \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ \pm 2\sqrt{\mu} & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Then eigenvalues of the above matrix are given by $\lambda = \frac{-1 \pm \sqrt{1 \pm 8\sqrt{\mu}}}{2}$.
Hence for $x > 0$ (i.e., $x = \sqrt{\mu}$), the eigenvalues are $\frac{-1 \pm \sqrt{1+8\sqrt{\mu}}}{2}$ and so the equilibrium point $(\sqrt{\mu}, 0)$ is a *saddle*; whilst for $x < 0$ (i.e., $x = -\sqrt{\mu}$), the eigenvalues are $\frac{-1 \pm \sqrt{1-8\sqrt{\mu}}}{2}$ and so the equilibrium point $(-\sqrt{\mu}, 0)$ is a *stable node*, which becomes a *stable spiral* for $\sqrt{\mu} > \frac{1}{8}$. however the immediate bifurcation as μ increases through zero is of saddle-node type. Figure 2.14 shows the saddle-node bifurcation for $\mu = 0.01$.

(b) **Transcritical bifurcation:**
In bifurcation theory, a field within mathematics, a transcritical bifurcation is a particular kind of local bifurcation, meaning that it is characterized by an equilibrium having an eigenvalue whose real part passes through zero. A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied. In other words, both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide. So the unstable fixed point becomes stable and vice versa.

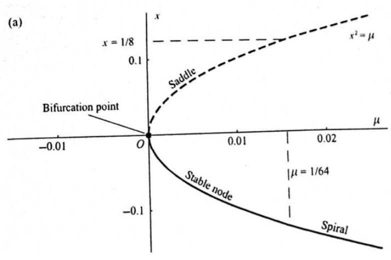

Figure 2.13: A saddle-node bifurcation

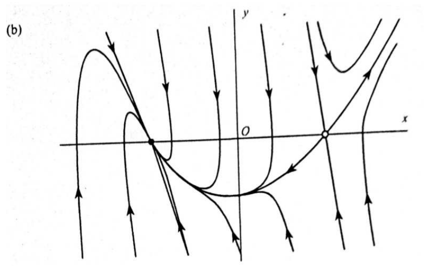

Figure 2.14: phase diagram for $\mu = 0.01$

Example 2.29 Consider the dynamical system defined by

$$\frac{dx}{dt} = rx - x^2, \text{ where } r, x \text{ are reals.} \tag{2.86}$$

Solution: Here $f(x) = rx - x^2$, $f'(x) = r - 2x$. This equation is similar to logistic equation but in this case we allow r and x to be positive or negative (while in the logistic equation x and r must be non-negative). The two fixed points are at $x = 0$ and $x = r$. When the parameter r is negative, the fixed point at $x = 0$ is stable and the fixed point $x = r$ is unstable. But for $r > 0$, the point at $x = 0$ is unstable and the point at $x = r$ is stable. So the bifurcation occurs at $r = 0$ (See in Figure 2.15).

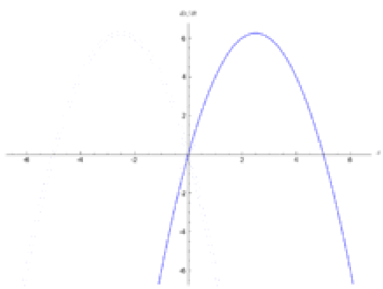

Figure 2.15: A normal transcritical bifurcation

Example 2.30 Consider the dynamical system defined by

$$\dot{x} = y, \; \dot{y} = \mu x - x^2 - y, \text{ where } \mu \text{ is real.} \tag{2.87}$$

Solution: It has equilibrium points at $(0, 0)$ and $(\mu, 0)$. Again all the equilibrium points lie only in the plane $y = 0$ in the (x, y, μ) space. The bifurcation curves are given by $x(x - \mu) = 0$, which are two straight lines intersecting at the origin(see Fig. 2.16).

There is a bifurcation point at $\mu = \mu_0 = 0$, since the number of equilibrium points changes from two ($\mu < 0$) to one ($\mu = 0$) and back to two ($\mu > 0$) as μ increases. Near the origin

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ \mu & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then eigenvalues of the above matrix are given by $\lambda = \frac{-1 \pm \sqrt{1+4\mu}}{2}$.
For $\mu < 0$, the origin is *stable*; a *node* if $-\frac{1}{4} < \mu < 0$, and a *spiral* if $\mu < -\frac{1}{4}$. If $\mu > 0$, then the origin is *unstable*.
Near $x = \mu$, with $x = \mu + x^{'}$ and $y = y^{'}$,

$$\begin{bmatrix} \dot{x}^{'} \\ \dot{y}^{'} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ -\mu & -1 \end{bmatrix} \begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix}.$$

Then eigenvalues of the above matrix are given by $\lambda = \frac{-1 \pm \sqrt{1-4\mu}}{2}$.
For $\mu < 0$, $x = \mu$ is a *unstable*; if $\mu > 0$ then $x = \mu$ is a *stable node* for $0 < \mu < \frac{1}{4}$ and a *stable spiral* for $\mu > \frac{1}{4}$.
This is an example of **transcritical bifurcation** where, at the intersection of two bifurcation curves, stable equilibrium switches from one curve to other at the bifurcation point. As μ increases through zero, the saddle point collides with the node at the origin, and then remains there whilst the stable node moves away from the origin.

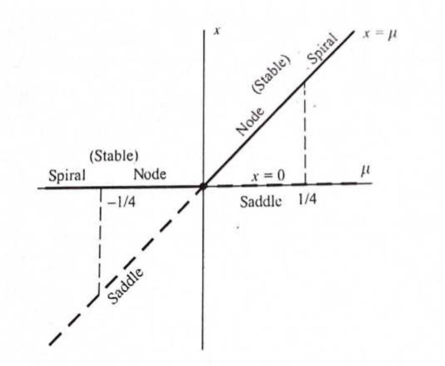

Figure 2.16: A transcritical bifurcation

(c) **Period-doubling or pitchfork or flip bifurcation:**
In mathematics, a period doubling bifurcation in a discrete dynamical system is a bifurcation in which a slight change in a parameter value in the system's equations leads to the system switching to a new behavior with twice the period of the original system. With the doubled period, it takes twice as many iterations as before for the numerical values visited by the system to repeat themselves. A period doubling cascade is a sequence of doubling and further doublings of the repeating period, as the parameter is adjusted further and further. Period doubling bifurcations can also occur in continuous dynamical systems, namely when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one.

Formal definition: An ODE

$$\dot{x} = f(x, r)$$

described by a one parameter function f with parameter $r \in \Re$ satisfying:

$$-f(x, r) = f(-x, r) \text{ (f is an odd function)}$$

$$\frac{\partial f}{\partial x}(0, r_o) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0, r_o) = 0, \quad \frac{\partial^3 f}{\partial x^3}(0, r_o) \neq 0,$$
$$\frac{\partial f}{\partial r}(0, r_o) = 0, \quad \frac{\partial^2 f}{\partial r \partial x}(0, r_o) \neq 0.$$

has a pitchfork bifurcation at $(x, r) = (0, r_o)$. The form of the pitchfork is given by the sign of the third derivative:

$$\frac{\partial^3 f}{\partial x^3}(0, r_o) \begin{cases} < 0, & \text{supercritical} \\ > 0, & \text{subcritical} \end{cases}$$

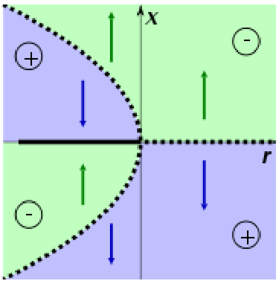

Figure 2.17: A subcritical case of pitchfork bifurcation

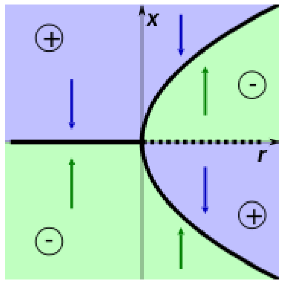

Figure 2.18: A supercritical case of pitchfork bifurcation

Example 2.31 The normal form for the **supercritical case**(See in Figure 2.30) is

$$\frac{dx}{dt} = rx - x^3, \text{ where } r \text{ is real.} \tag{2.88}$$

Solution: For negative values of r, there is one stable equilibrium at $x = 0$. For $r > 0$ there is an unstable equilibrium at $x = 0$ and two stable equilibria at $x = \pm\sqrt{r}$.

Example 2.32 The normal form for the **subcritical case**(See in Figure 2.17) is

$$\frac{dx}{dt} = rx + x^3. \tag{2.89}$$

Solution: Here $f(x) = rx + x^3$, $f'(x) = r + 3x^3$, so for $r < 0$ the equilibrium at $x = 0$ is stable, and there are two unstable equilibria at $x = \pm\sqrt{-r}$. For $r > 0$ the equilibrium at $x = 0$ is unstable.

Example 2.33 Consider a dynamical system

$$\dot{x} = y, \ \dot{y} = \mu x - x^3 - y. \tag{2.90}$$

Solution: The (x, μ) bifurcation diagram, has a bifurcation point at $\mu = \mu_0 = 0$. Near the origin

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ \mu & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then eigenvalues of the above matrix are given by $\lambda = \frac{-1 \pm \sqrt{1+4\mu}}{2}$.
For $\mu < 0$ the origin is *stable*; a *node* if $-\frac{1}{4} < \mu < 0$, and a *spiral* if $\mu < -\frac{1}{4}$. If $\mu > 0$ then the origin

is *unstable*.
The additional equilibrium points are at $x = \pm\mu$ for $\mu > 0$. Linearization about these points shows that $x = \pm\mu$ are *stable nodes* for $0 < \mu < \frac{1}{8}$, and *stable spirals* for $\mu > \frac{1}{8}$. As μ increases through zero the stable node for $\mu < 0$ bifurcates into two stable nodes and a saddle point for $\mu > 0$. This is an example of **pitchfork bifurcation** (See in figure 2.19) named after its shape in the (x, μ) plane.
Pitchfork bifurcations are often associated with **symmetry breaking**. In the system above, $x = 0$ is the only (stable) equilibrium point for $\mu < 0$, but as μ increases through zero the system could be disturbed into either stable mode, thus destroying symmetry.

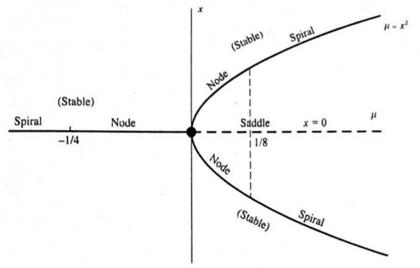

Figure 2.19: A pitchfork bifurcation

(d) **Hopf bifurcation:**
In the mathematical theory of bifurcations, a Hopf or PoincarAndronovHopf bifurcation, named after Henri Poincar, Eberhard Hopf, and Aleksandr Andronov, is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small-amplitude limit cycle branching from the fixed point.

Oscillation: Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. The term 'vibration' is precisely used to describe mechanical oscillation but used as a synonym of 'oscillation' too. Familiar examples include a swinging pendulum and alternating current power. Oscillations occur not only in mechanical systems but also in dynamic systems in virtually every area of science: for example the beating human heart, business cycles in economics, predator-prey population cycles in ecology, geothermal geysers in geology, vibrating strings in musical instruments, periodic firing of nerve cells in the brain, and the periodic swelling of Cepheid variable stars in astronomy.

Damped Oscillation: Any oscillation in which the amplitude of the oscillating quantity decreases with time. Also known as damped vibration.

Some bifurcations generate limit cycles or other periodic solutions. Consider the system

$$\dot{x} = \mu x + y - x(x^2 + y^2), \tag{2.91}$$
$$\dot{y} = -x + \mu y - y(x^2 + y^2) \tag{2.92}$$

where μ is the bifurcation parameter. The system has a single equilibrium point, at the origin. In polar co-ordinate the equations become

$$\dot{r} = r(\mu - r^2), \ \dot{\theta} = -1.$$

If $\mu \leq 0$ then the entire diagram consists of a stable spiral. If $\mu > 0$ then there is an unstable spiral at the origin surrounded by a stable limit cycle which grows out of thew origin- the steps in its development are shown in Fig. 2.20. This is an example of a Hopf bifurcation which generates a limit cycle. Typically for such cases the linearization of (2.91) and (2.92) predicts a centre at the origin, which proves proves to be incorrect: the origin changes from being asymptotically stable to being unstable without passing through the stage of being a centre.
The following is a simple version, restricted polar type equations, of a more general result.

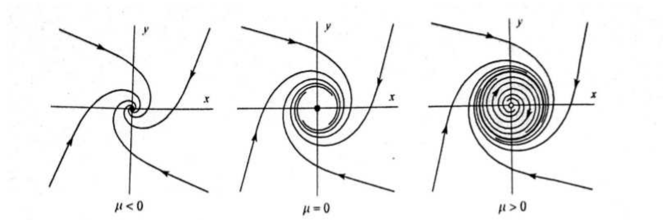

Figure 2.20: Developement of limit cycle in a Hopf bifurcation

Theorem 2.10 Given the equations $\dot{x} = \mu x + y - x f(r)$, $\dot{y} = -x + \mu y - y f(r)$, where $r = \sqrt{(x^2 + y^2)}$, $f(r)$ and $f^{'}(r)$ are continuous for $r \geq 0$, $f(0) = 0$, and $f(r) > 0$ for $r > 0$. The origin is the only equilibrium point. Then
(i) for $\mu < 0$ the origin is a stable spiral covering the whole plane;
(ii) for $\mu = 0$ the origin is a stable spiral;
(iii) for $\mu > 0$ there is a stable limit cycle whose radius increases from zero as μ increases from zero.

Example 2.34 Show that the equation

$$\ddot{x} + (x^2 + \dot{x}^2 - \mu)\dot{x} + x = 0$$

exhibits a Hopf bifurcation as μ increases through zero.

We can apply Theorem 2.10 to show that the system has no periodic solution for $\mu < 0$. Write the equation as

$$\dot{x} = X(x, y) = y, \qquad \dot{y} = Y(x, y) = -(x^2 + y^2 - \mu)y - x. \tag{2.93}$$

Then, $$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -x^2 - 3y^2 + \mu < 0$$

for all (x, y) and $\mu < 0$. Hence by Theorem , there can be no closed paths in the phase plane. For the critical case $\mu = 0$, we can use the Lyapunov function

$$V(x, y) = \frac{1}{2}(x^2 + y^2)$$

which results in $\dot{V}(x, y) = x\dot{x} + y\dot{y} = xy + y(-(x^2 + y^2 - \mu)y - x) = -(x^2 + y^2)y^2, \;\; (\because \mu = 0).$

Since $\dot{V}(x, y)$ vanishes along the line $y = 0$, but is otherwise strictly negative, the origin is an asymptotically stable equilibrium point. For $\mu > 0$ the system has a stable limit cycle with path $x^2 + y^2 = \mu$ which evidently emerges from the origin at $\mu = 0$ with a radius which increases with μ.

Example 2.35 Consider the dynamical system defined by the two equations

$$\frac{dx}{dt} = -y + x(a - x^2 - y^2), \qquad \frac{dy}{dt} = x + y(a - x^2 - y^2)$$

for real x, y, a.

Solution: There is a trivial steady state at $x = y = 0$. To examine its linear stability, we write $x = 0 + \tilde{x}, y = 0 + \tilde{y}$. Substituting this into the above defining equations and linearising, we get

$$\frac{d\tilde{x}}{dt} = -\tilde{y} + a\tilde{x}, \qquad \frac{d\tilde{y}}{dt} = \tilde{x} + a\tilde{y}$$

Then the eigenvalue of the above linearized system are $\lambda = a \pm i$. Now, for $a > 0$ then Real part of $\{\lambda\} > 0$ and $|\tilde{x}|, |\tilde{y}|$ both tends to infinity as t tends to infinity (linear instability).
Also, for $a < 0$ then Real part of $\{\lambda\} < 0$ and and $|\tilde{x}|, |\tilde{y}|$ both tends to zero as t tends to infinity (linear stability).
The fact that λ is complex confers a new dynamical feature not encountered in the previous examples: that of temporal oscillation. For $a < 0$, for example, the progress of $\tilde{x}$ and $\tilde{y}$ in towards the origin is via a damped oscillation, as sketched in the left hand plot in Fig. 2.21 , rather than a straightforward exponential decay. As in the other bifurcation examples, the loss of stability

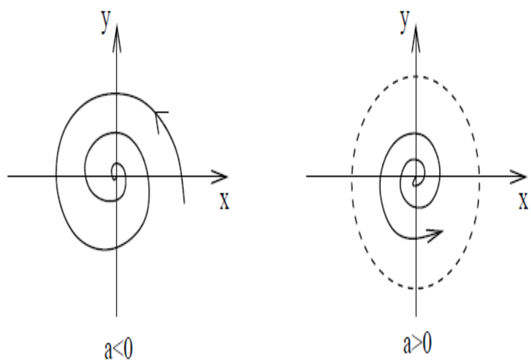

Figure 2.21: Solution curve for $a < 0$ and $a > 0$.

at $a = 0$ gives rise to a new solution for $a > 0$. In this case, the new solution is periodic:

$$x = \sqrt{a}\cos(t + t_0), y = \sqrt{a}sin(t + t_0)$$

The system orbits round the limit cycle drawn by the dashed line in the right hand sketch above. The bifurcation diagram in Fig. 2.22 is then as follows.

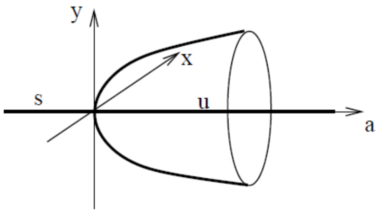

Figure 2.22: Hopf-bifurcation.

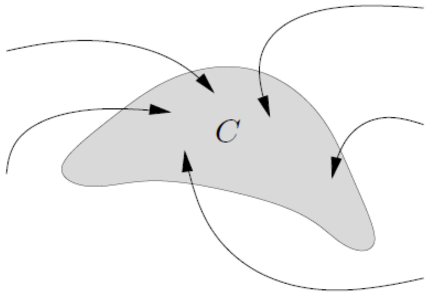

Figure 2.23: Invariant set.

Invariant Set: We consider autonomous, time-invariant nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ a set $C \subseteq \Re^n$ is invariant (*w.r.t.* system, or $\mathbf{f}$) if for every trajectory x,

$$\mathbf{x}(t) \in C \Rightarrow \mathbf{x}(\tau) \in C \text{ for all } \tau \geq t$$

if trajectory enters C, or starts in C, it stays in C
trajectories can cross into boundary of C, but never out of C. An invariant set have been shown graphically in Fig. 2.23.

(ii) Global Bifurcation:

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence 'global'). Global bifurcations are divided into many parts such as

(a) Homoclinic bifurcation in which a limit cycle collides with a saddle point.
(b) Heteroclinic bifurcation in which a limit cycle collides with two or more saddle points.
(c) Infinite-period bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.

(a) **Homoclinic bifurcation:**
In mathematics, a homoclinic bifurcation is a global bifurcation which often occurs when a periodic orbit collides with a saddle point. The image below shows a phase portrait before, at, and after a homoclinic bifurcation in 2 D. The periodic orbit grows until it collides with the saddle point. At the bifurcation point the period of the periodic orbit has grown to infinity and it has become a homoclinic orbit. After the bifurcation there is no longer a periodic orbit. A Homoclinic bifurcation occurs when a periodic orbit collides with a saddle point. In Fig.- 2.24, **Left panel:** for small parameter values, there is a saddle point at the origin and a limit cycle in the first quadrant. **Middle panel:** as the bifurcation parameter increases, the limit cycle grows until it exactly intersects the saddle point, yielding an orbit of infinite duration. **Right panel:**

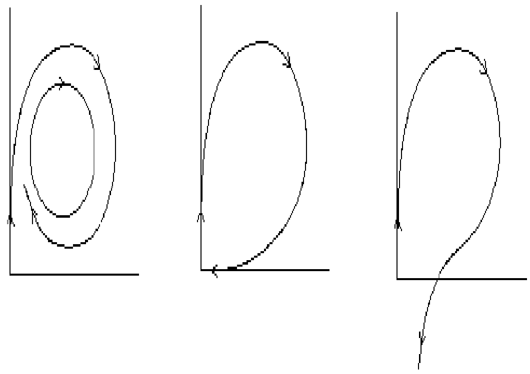

Figure 2.24: Homoclinic bifurcation.

when the bifurcation parameter increases further, the limit cycle disappears completely.

(b) **Heteroclinic bifurcation**
In mathematics, particularly dynamical systems, a heteroclinic bifurcation is a global bifurcation involving a heteroclinic cycle.

(c) **Infinite period bifurcation:**
In mathematics, an infinite-period bifurcation is a global bifurcation that can occur when two fixed points emerge on a limit cycle. As the limit of a parameter approaches a certain critical value, the speed of the oscillation slows down and the period approaches infinity. The infinite-period bifurcation occurs at this critical value. Beyond the critical value, the two fixed points emerge continuously from each other on the limit cycle to disrupt the oscillation and form two saddle points.

2.13 Worked Out Examples

Example 2.36 Find the eigenvalues for the system and show that the system is stable

$$\frac{dy_1}{dt} = -y_1 - 2y_2, \qquad \frac{dy_2}{dt} = 2y_1 - y_2 \tag{2.94}$$

Sketch the phase diagram.

Solution: Here

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

and the eigenvalues are $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$. Both eigenvalues are complex with negative real part, so the system is stable. Also the phase diagram is shown in Fig.-2.25

Example 2.37 Investigate the stability of the zero solution of the system $\dot{x} = -x - 2y^2$, $\dot{y} = xy - y^3$.

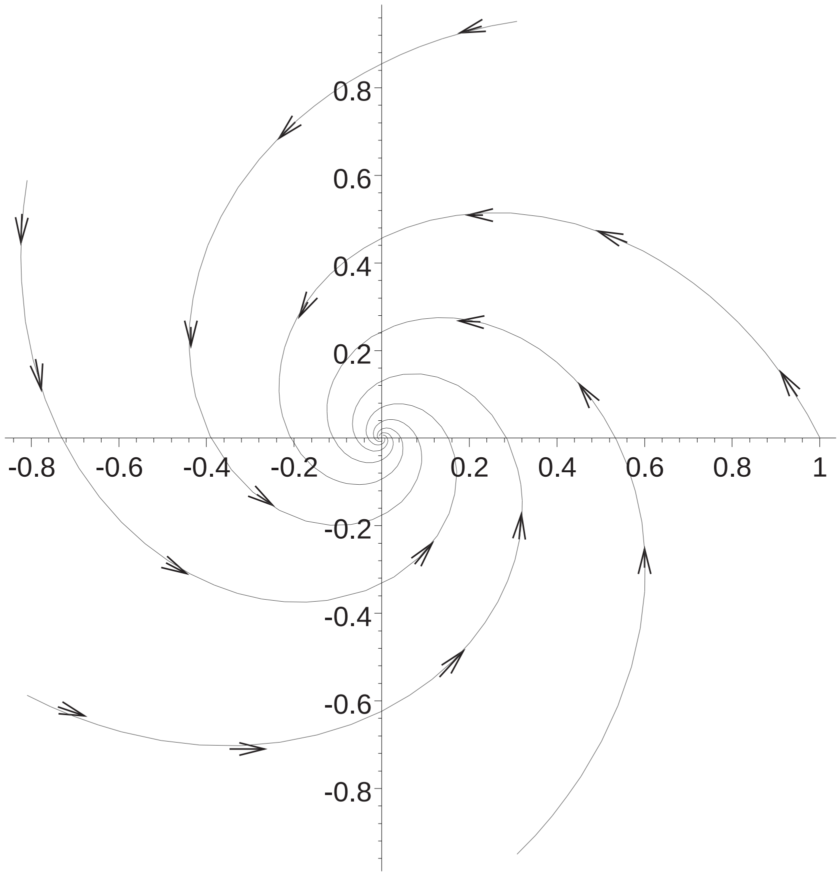

Figure 2.25: The phase portrait of the differential equation(2.94)

Solution: Since $\dot{x} = 0$ and $\dot{y} = 0$ at $(0, 0)$, so origin is an equilibrium point.
Let us consider a family of curves

$$\begin{aligned} V(x, y) &= x^2 + 2y^2 = \alpha, \;\; 0 < \alpha < \infty \\ \text{Then } \dot{V}(x, y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 2x(-x - 2y^2) + 4y(xy - y^3)2y = -2(x^2 + 2y^4) \end{aligned}$$

which is negative everywhere except at the origin. Therefore we have found a strong Lyapunov function for the system. Hence the zero solution is the uniformly and asymptotically stable in the Lyapunov sense.

Example 2.38 By linearizing around the critical points, sketch the phase space trajectories and check the stability of

$$y'' = -y' + y - y^2 \tag{2.95}$$

Solution: Putting $y_1 = y$ and $y_2 = \dot{y_1}$ in (2.95), then the said system becomes

$$y_1' = y_2, \qquad y_2' = -y_2 + y_1(1 - y_1). \tag{2.96}$$

This has two same critical points, one at $(y_1, y_2) = (0, 0)$ and the second at $(y_1, y_2) = (1, 0)$. Near $(0, 0)$ the system linearizes to the system

$$y_1' = y_2, \qquad y_2' = y_1 - y_2. \tag{2.97}$$

Solving the said system of differential equations, we have two eigenvalues $\lambda_1 = -(1+\sqrt{5})/2$, $\lambda_2 = (-1+\sqrt{5})/2$. Here $\lambda_2 = (-1+\sqrt{5})/2$ is positive, so the point (0, 0) is saddle point. For $\lambda_1 = -(1+\sqrt{5})/2$, the eigenvector is

$$\mathbf{y}_{11} = \begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix} \tag{2.98}$$

and for $\lambda_2 = (-1+\sqrt{5})/2$, the eigenvector is

$$\mathbf{y}_{21} = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix} \tag{2.99}$$

Also, one eigenvalue is positive at $(0, 0)$, so the system is unstable at $(0, 0)$ by Theorem 2.2. Using section (2.11.1) near $(1, 0)$, we write $y_1 = 1 + \eta$ and get

$$\eta' = y_2, \qquad y_2' = -y_2 + y_1 - y_1^2 = -y_2 + 1 + \eta - (1+\eta)^2 \approx -\eta - y_2 \tag{2.100}$$

so the eigenvalues are

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \tag{2.101}$$

As the real part of all eigenvalues are negative at $(1, 0)$, so the system is stable at $(1, 0)$ by Theorem 2.2. Also the phase portrait of the given dynamical system is presented in Fig.-2.26 which also show the stability of the said dynamical system.

Example 2.39 Show that the given system of linear homogenous differential equations is unstable and draw the phase diagram of the differential equations.

$$\frac{dy_1}{dt} = -3y_1 + 2y_2, \qquad \frac{dy_2}{dt} = -2y_1 + 2y_2 \tag{2.102}$$

Solution: The solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \tag{2.103}$$

Now draw the phase diagram for this solution and name the type of stationary point (saddle point or outward improper). So, any point that starts on the

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.104}$$

eigenvector will move inwards, since $c_1 = 0$ and $c_2 e^{-2t}$ gets small as t increases, anywhere on the other eigenvectors will move straight outwards. If you aren't on either eigenvector, the amount along the negative eigenvalue eigenvector decreases and the amount along the positive eigenvector eigenvalue increases and so you move outwards getting closer and closer to the positive eigenvalue line. Here one eigenvalue is positive, so the stationary point is a saddle point. The phase diagram is shown in Fig.-2.27

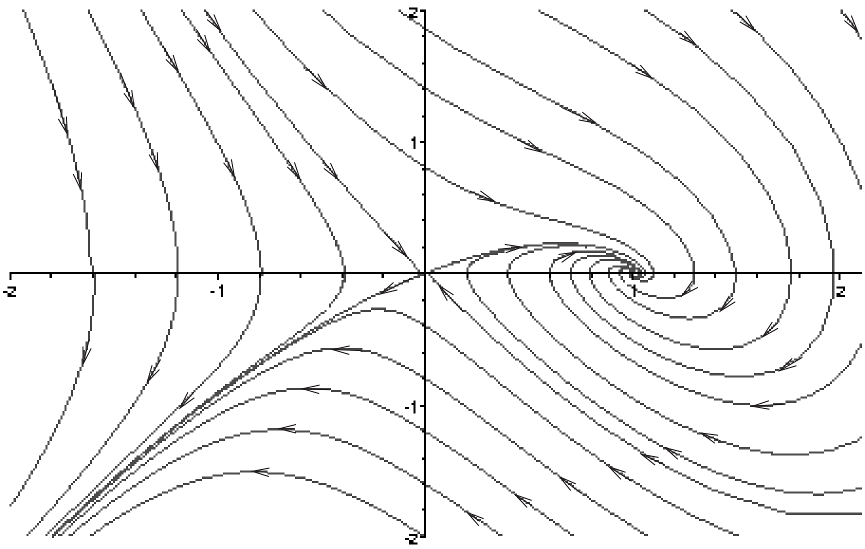

Figure 2.26: The phase portrait of the differential equation(2.95)

Example 2.40 Find the general solutions and the fundamental matric for the system.

$$\frac{dy_1}{dt} = 2y_1 - y_2, \qquad \frac{dy_2}{dt} = -4y_2 \tag{2.105}$$

Sketch the phase diagram and and describe the stationary point.

Solution: Here

$$A = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix}$$

and the spectrum[1] is ($\lambda_1 = 2$, $\lambda_2 = -4$).
For $\lambda_1 = 2$, corresponding eigenvector is

$$\mathbf{y}_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\lambda_2 = -4$, corresponding eigenvector is

$$\mathbf{y}_{21} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

So the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-4t}.$$

As $\lambda_1 \neq \lambda_2$, so the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{-4t} \\ 0 & 6e^{-4t} \end{pmatrix}$$

[1]The set of eigenvalues of a matrix is sometimes called its spectrum

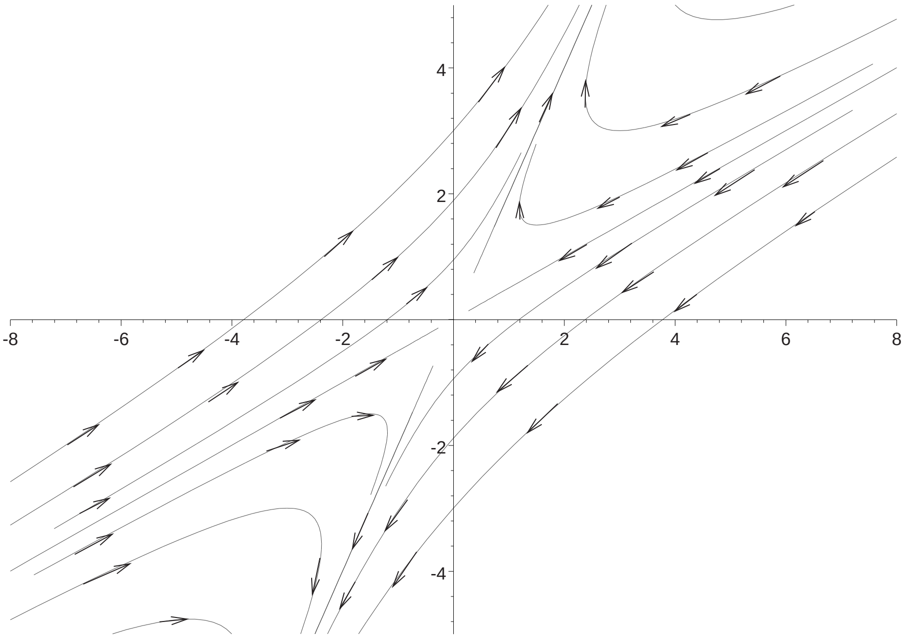

Figure 2.27: The phase portrait of the differential equation(2.102)

. As one eigenvalue is positive, so the system is unstable at the saddle point $(0, 0)$ by Theorem 2.2. Also the phase portrait of the given dynamical system is presented in Fig.-2.28 which also show the unstable graph of the said dynamical system.

Example 2.41 Show that the given system of linear homogenous differential equations is unstable and find the fundamental matrix of the system. Draw the phase diagram of the differential equations.

$$\frac{dy_1}{dt} = 3y_1 + y_2, \qquad \frac{dy_2}{dt} = y_1 + 3y_2 \tag{2.106}$$

Solution: Here

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$. Both eigenvalues are positive, so the system is unstable by Theorem 2.2.
For $\lambda_1 = 2$, corresponding eigenvector is

$$\mathbf{y}_{11} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and $\lambda_2 = 4$, corresponding eigenvector is

$$\mathbf{y}_{21} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

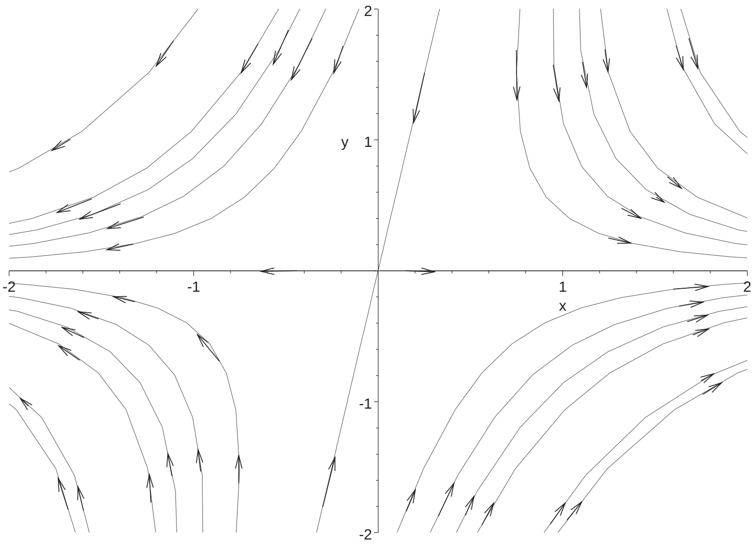

Figure 2.28: The phase portrait of the differential equation(2.105)

So the general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \tag{2.107}$$

As $\lambda_1 \neq \lambda_2$, so the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}.$$

Also the phase portrait of the given dynamical system in (2.106) is presented in Fig.-2.29 which also show the unstable graph of the said dynamical system.

Example 2.42 Find the bifurcation points of the system $\dot{x} = -\eta x + y, \dot{y} = -\eta x - 3y$.

Solution: Let $x = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A(\eta) = \begin{pmatrix} -\eta & 1 \\ -\eta & -3 \end{pmatrix}$

so that, in matrix form, the system is equivalent to

$$\dot{x} = A(\eta)x$$

If $\eta \neq 0$, the system has only one equilibrium point, at the origin; if $\eta = 0$, equilibrium occurs at all points on $y = 0$. We find the eigenvalues λ of $A(\eta)$ from

$$| A(\eta) - \lambda I |= 0$$

$$\Rightarrow \quad \begin{vmatrix} -\eta - \lambda & 1 \\ -\eta & -3 - \lambda \end{vmatrix} = 0$$

Hence λ satisfies

$$\lambda^2 + (3 + \eta)\lambda + 4\eta = 0$$

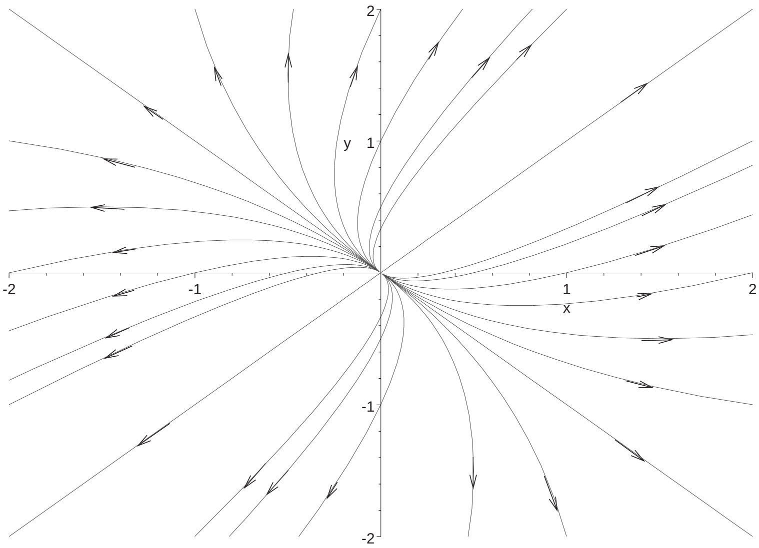

Figure 2.29: The phase portrait of the differential equation(2.106)

which has the roots

$$\lambda_1,\ \lambda_2 = \frac{1}{2}[-\eta - 3 \pm \sqrt{(\eta-1)(\eta-9)}]$$

we look for bifurcations where the character of the root changes- having regard to whether they are real or complex, and to the sign when they are real or the sign of the real part when they are complex.
The classification of the origin is:

η	λ_1, λ_2	Type
$\eta < 0$	Real, opposite signs	Saddle
$0 < \eta < 1$	Real, negative	Stable node
$1 < \eta < 9$	Complex, negative real part	Stable spiral
$\eta > 9$	Real negative	Stable node

The system has a single bifurcation point $\eta = 0$, where there is a change from a stable node to a saddle.

Example 2.43 Consider the dynamical system defined by

$$\frac{dx}{dt} = a - x^2, \text{ where } a \text{ is real.}$$

Solution: A steady state solution ($\frac{dx}{dt} = 0$) of this system is given by $x = x^* = \pm\sqrt{a}$. Therefore, for $a < 0$ we have no real solution and for $a > 0$ we have two real solution. We now consider each of the two solutions for $a > 0$, and examine their linear stability in the usual way. First, we add a small perturbation:
$x = x^* + \tilde{x}$. Substituting this value in the above equation we get

$$\frac{d\tilde{x}}{dt} = (a - x^{*2}) - 2x^*\tilde{x} - \tilde{x}^2$$

The value of the bracket in the RHS is zero, from the definition of equilibrium point. At first order in the perturbation, $\tilde{x}$ we therefore have $\frac{d\tilde{x}}{dt} = -2x^*\tilde{x}$ and its solution becomes $\tilde{x}(t) = A\exp(-2x^*t) = Ae^{-2x^*t}$.
Then for $x^* = +\sqrt{a}$, $|\tilde{x}(t)| \to 0$ as $t \to \infty$ (linear stability).
Again, for $x^* = -\sqrt{a}$, $|\tilde{x}(t)| \to \infty$ as $t \to \infty$ (linear instability).
As sketched in the bifurcation diagram below in Fig. 2.9, therefore, the saddlenode bifurcation at $a = 0$ corresponds to the creation of two new solution branches. One of these is linearly stable, the other linearly unstable.

Example 2.44 Investigate the bifurcation of the system

$$\dot{x} = X(x, y) = 2x(\mu - x) - (x + 1)y^2, \ \ \dot{y} = Y(x, y) = y(x - 1).$$

Solution: Equilibrium occurs where

$$2x(\mu - x) - (x + 1)y^2 = 0, \ \ y(x - 1) = 0,$$

which can be separate into three curves in (x, y, μ) space:

$$C_1 : x = 0, \ y = 0; \ C_2 : x = \mu, \ y = 0; \ C_3 : x = 1, \ y^2 = \mu - 1 (\mu \geq 1).$$

The equilibrium states are shown in Fig. 2.30. The Jacobian of $[X(x, y), Y(x, y)]^T$ is

$$J(x, y, \mu) = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix} = \begin{bmatrix} 2\mu - 4x - y^2 & -2(x + 1)y \\ y & x - 1 \end{bmatrix}$$

We need to investigate the eigenvalues of $J(x, y, \mu)$ in each equilibrium state since this will decide the type of each linear approximation.
(a) Equilibrium points on C_1. Here

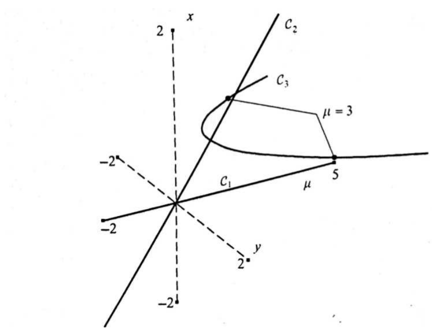

Figure 2.30: Equilibrium curves for $\dot{x} = 2x(\mu - x) - (x + 1)y^2, \ \dot{y} = y(x - 1)$.

$$J(0, 0, \mu) = \begin{bmatrix} 2\mu & 0 \\ 0 & -1 \end{bmatrix}$$

and the eigenvalues of $J(0, 0, \mu)$ are given by

$$\det\left[J(0, 0, \mu) - mI_2\right] = \begin{vmatrix} 2\mu - m & 0 \\ 0 & -1 - m \end{vmatrix} = 0$$

so that $m = -1$ or 2μ. Hence for $\mu < 0$, the equilibrium point is a stable node and for $\mu > 0$ a saddle point.
(b) Equilibrium points on C_2. In this case

$$J(\mu, 0, \mu) = \begin{bmatrix} -2\mu & 0 \\ 0 & \mu - 1 \end{bmatrix}.$$

The eigenvalues of $J(\mu, 0, \mu)$ are $m = -2\mu$ or $\mu - 1$. Hence for $\mu < 0$ and $\mu > 1$, the eigenvalues are real and opposite signs, indicating a saddle, whilst for $0 < \mu < 1$ the eigenvalues are both real and negative indicating a stable node.

From (a) and (b) it follows that there is a transcritical bifurcation at $\mu = 0$. The situation at $\mu = 1$ is more complicated since curve C_3 intersects C_2 there.
(c) Equilibrium points on C_3. On this curve

$$J(1, \sqrt{(\mu-1)}, \mu) = \begin{bmatrix} \mu - 3 & -4\sqrt{(\mu-1)} \\ \sqrt{(\mu-1)} & 0 \end{bmatrix}$$

Hence the eigenvalues are given by

$$\begin{vmatrix} \mu - 3 - m & -4\sqrt{(\mu-1)} \\ \sqrt{(\mu-1)} & -m \end{vmatrix} = 0 \text{ or, } m^2 - (\mu - 3)m + 4(\mu - 1) = 0.$$

The eigenvalues can be defined as

$$m_1, m_2 = \frac{1}{2}[\mu - 3 \pm \sqrt{\{(\mu - 11)^2 - 96\}}]$$

The eigenvalues for $\mu > 1$ are of the following types
(I) $1 < \mu < 11 - 4\sqrt{6}(\approx 1.202)$: m_1, m_2 real and negative(stable node);
(II) $11 - 4\sqrt{6} < \mu < 3$: m_1, m_2 complex conjugates with negative real part (stable spiral);
(III)$3 < \mu < 11 + 4\sqrt{6}(\approx 20.798)$: m_1, m_2 complex conjugates with positive real part (unstable spiral);
(IV)$11 + 4\sqrt{6} < \mu$: m_1, m_2 both real and positive (unstable node).
Results (b) and (c) (I) above show that a pitchfork bifurcation occurs at the bifurcation point at $\mu = 1$. As μ increases through $\mu = 3$,the stable spirals at $x = 1, y = \sqrt{2}$ become unstable spirals. Hence $\mu = 3$ is a bifurcation point indicating a change of stability. A similar bifurcation occurs at the symmetric point $x = 1, y = -\sqrt{(\mu - 1)}$. These are often referred to as Hopf bifurcation: as we shall see in the next section this stability change of a spiral can generate a limit cycle under some circumstances, but not in this example.
To summarize, the system has three bifurcation points: a transcritical bifurcation at $\mu = 0$, a pitchfork bifurcation at $\mu = 1$ and Hopf bifurcation at $\mu = 3$.

2.14 Multiple Choice Questions

1. Consider the system of ODE $\frac{dY}{dx} = AY,\ Y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$
   (a) $y_1(x) \to \infty$ and $y_2(x) \to 0$ as $x \to \infty$ (b) $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$
   (c) $y_1(x) \to \infty$ and $y_2(x) \to -\infty$ as $x \to -\infty$ (d) $y_1(x) \to \infty$ and $y_2(x) \to -\infty$ as $x \to -\infty$
   Ans. (a) and (c). **NET(MS): (June)2012**

2. Let $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ satisfy $\frac{dY}{dx} = AY$, $t > 0$, $Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where A is a 2×2 constant matrix with real entries satisfying $trace\, A = 0$ and $det\, A > 0$. Then $y_1(x)$ and $y_2(x)$ both are
(a) monotonically decreasing functions of t. (b) monotonically increasing functions of t.
(c) oscillating functions of t. (d) constant functions of t.
Ans. (c). **NET(MS): (Dec.)2012**

3. Consider the system of ODE in $\Re^2$, $\frac{dY}{dt} = AY$, $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $t > 0$ where $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. Then **NET(MS): (Dec.)2015**
(a) $y_1(t)$ and $y_2(t)$ are monotonically increasing for $t > 0$.
(b) $y_1(t)$ and $y_2(t)$ are monotonically increasing for $t > 1$.
(c) $y_1(t)$ and $y_2(t)$ are monotonically decreasing for $t > 0$.
(d) $y_1(t)$ and $y_2(t)$ are monotonically decreasing for $t > 1$.
Ans. (d).

4. The critical point of the system $\frac{dx}{dt} = -4x - y$, $\frac{dy}{dt} = x - 2y$ is an **NET(MS): (June)2015**
(a) asymptotically stable node (b) unstable node
(c) asymptotically stable spiral (d) unstable spiral.
Ans. (a).
Hint. The eigenvalues are $-3, -3$. So the eigenvalues are real and negatives. Hence the critical points of the system is an asymptotically stable node.

5. The critical point $(0, 0)$ of the dynamical system $\frac{dx}{dt} = 2x - 7y$, $\frac{dy}{dt} = 3x - 8y$ is an **NET(MS): (June)2018**
(a) asymptotically stable node (b) unstable node
(c) asymptotically stable spiral (d) unstable spiral.
Ans. (a).
Hint. The eigenvalues are $-1, -5$. So the eigenvalues are real and negatives. Hence the critical points of the system is an asymptotically stable node.

6. Let $\frac{d^2y}{dx^2} - q(x)y = 0$, $0 \le x < \infty$ with $y(0) = y'(0) = 0$, where $q(x)$ is positive monotonically increasing continuous function. Then, **NET(MS): (Dec.)2011**
(a) $y(x) \to \infty$ as $x \to \infty$ (b) $y'(x) \to \infty$ as $x \to \infty$
(c) $y(x)$ has finitely many zeros in $[0, \infty)$ (d) $y(x)$ has infinitely many zeros in $[0, \infty)$
Ans. (a) and (b).

7. Let $(x(t), y(t))$ satisfy the system of ODEs
$\frac{dx}{dt} = -x + ty$
$\frac{dy}{dt} = tx - y$
If $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions and $\Phi(t) = x_1(t)y_2(t) - x_2(t)y_1(t)$ then $\frac{d\Phi}{dt}$ is equal to
1. -2Φ. 2. 2Φ. 3. $-\Phi$. 4. Φ **[NET-DEC-2016]**
Ans: 1.

8. Test the stability of the system $\dot{x} = y + \frac{xy}{1+t^2}$, $\dot{y} = -x - y + \frac{y^2}{1+t^2}$
(A) Stable (B) Asymptotically stable (C) Unstable (D) Quasi stable
Ans. (A), (B) and (D)

2.15 Review Exercises

1. Prove that the linear autonomous plane systems $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ is stable if $p^2-4q > 0,\ q > 0,\ p < 0$ and unstable if $p^2-4q > 0,\ q > 0,\ p > 0$ where $p = a+d,\ q = ad-bc$.

2. For the system $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ where $ad - bc = 0$, show that all points on the line $cx + dy = 0$ are equilibrium points. Sketch the phase diagram for the system $\dot{x} = x - 2y$ and $\dot{y} = 3x - 6y$.
3. Find a fundamental matrix for the system
$$\frac{dy_1}{dt} = y_1 + y_2, \qquad \frac{dy_2}{dt} = y_1 + y_3, \qquad \frac{dy_3}{dt} = y_3.$$
Sketch the phase diagram.
Ans. $\Phi(t) = \begin{pmatrix} e^t & te^t & \frac{t^2e^t}{2} \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}.$
4. Find a fundamental matrix for the system
$$\frac{dy_1}{dt} = y_1 + y_2, \qquad \frac{dy_2}{dt} = y_2, \qquad \frac{dy_3}{dt} = y_3.$$
Sketch the phase diagram.
Ans. $\Phi(t) = \begin{pmatrix} e^t & 0 & te^t \\ 0 & 0 & e^t \\ 0 & e^t & 0 \end{pmatrix}.$
5. Find the fundamental matric and the solution $x(t)$ such that $x(0) = [3\ \ 1]^T$ for the system.
$$\frac{dx_1}{dt} = x_1 - 2e^{-t}x_2, \qquad \frac{dx_2}{dt} = e^t - x_2.$$
Ans. $\Phi(t) = \begin{pmatrix} 2 & e^{-t} \\ e^t & 1 \end{pmatrix}$ and $x(t) = \begin{pmatrix} 4 - e^{-t} \\ 2e^{-t} - 1 \end{pmatrix}$
6. Show that the system
$$\ddot{x} + (2 + 3x^2)\dot{x} + x = 0$$
is equivalent to the first-order system $\dot{x} = y - x^3,\ \dot{y} = -x + 2x^3 - 2y$. Show that the origin in the (x, y) plane is asymptotically stable.
7. Find the equilibrium points of the nonlinear system
$$\frac{dx_1}{dt} = x_1^2 - x_2 + x_3, \qquad \frac{dx_2}{dt} = x_1 - x_2, \qquad \frac{dx_3}{dt} = 2x_2^2 + x_3 - 2$$
Ans. $(-2, -2, -6)$ and $(1, 1, 0)$.
8. Show that all solutions of $\ddot{x} + \left\{a + c(1 + t^2)^{-1}\right\}x = f(t)$ are stable if $a > 0$.
9. Show that all solutions of $\ddot{x} + v(x^2 - 1)\dot{x} + x = 0$ has an asymptotically stable zero solution if $v < 0$.
10. Show that all solutions of $\ddot{x} + k\dot{x} + \sin x = 0$ is an asymptotically stable for $k > 0$.

11. Find the linear approximations at the equilibrium points of the nonlinear system

$$\frac{dx_1}{dt} = x_1^2 - x_2 + x_3, \qquad \frac{dx_2}{dt} = x_1 - x_2, \qquad \frac{dx_3}{dt} = 2x_2^2 + x_3 - 2.$$

Ans. At $(1,1,0)$, the linear approximation is $\dot{\xi} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \xi.$

and at $(-2,-2,-6)$, the linear approximation is $\dot{\xi} = \begin{pmatrix} -4 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & -8 & 1 \end{pmatrix} \xi.$

12. Show that all solutions of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -2 + (1+t^2)^{-1} & 0 & -1 \\ -1 & -1 + (1+t^2)^{-1} & -1 \\ -1 & 1 & -3 + (1+t^2)^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

are asymptotically stable.
Hint. Use the Theorem 2.9.

13. Show that every solution of the system $\dot{x} = -t^2 x, \ \dot{y} = -ty$ is asymptotically stable.

14. Show that the coefficient matrix of the linearization $\frac{dx}{dt} = -2x, \ \frac{dy}{dt} = -4y$ of the dynamical system

$$\begin{aligned} \frac{dx}{dt} &= x^2 - 2x - xy \\ \frac{dy}{dt} &= y^2 - 4x + xy, \end{aligned}$$

at the point $(0,0)$ has the negative eigenvalues $\lambda_1 = -2, \ \lambda_2 = -4$. Also sow that $(0,0)$ is a nodal sink for said dynamical system.

15. Show that all solutions of

$$\begin{aligned} \frac{dx_1}{dt} &= t^{-2}x_1 - 4x_2 - 2x_3 + t^{-2} \\ \frac{dx_2}{dt} &= -x_1 + t^{-2}x_2 + x_3 + \sin t \\ \frac{dx_3}{dt} &= t^{-2}x_1 - 9x_2 - 4x_3 + 1 \end{aligned}$$

are asymptotically stable.
Hint. Use the Theorem 2.9.

16. Show that all solutions of

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + e^{-t}x_2 - 3x_3 + e^2 \\ \frac{dx_2}{dt} &= -2x_1 + e^{-t}x_2 + x_3 + 1 \\ \frac{dx_3}{dt} &= (4 + e^{-t})x_1 - x_2 - 4x_3 + e^{-t} \end{aligned}$$

are asymptotically stable.
Hint. Use the Theorem 2.9.

17. The differential equation $\frac{dx}{dt} = \frac{x(10-x)}{10} - h$ models a logistic population with harvesting at rate h. Determine the dependence of number of critical points on the parameter h and then construct a bifurcation diagram.
Ans. There are two critical points if $h < 2.5$ and one critical point if $h = 2.5$ and no critical point if $h > 2.5$. The bifurcation diagram is the parabola $(c-5)^2 = 25 - 10h$ in the $hc-$plan.

18. The differential equation $\frac{dx}{dt} = \frac{x(x-5)}{100} + s$ models a population with stocking at rate s. Determine the dependence of number of critical points c on the parameter h and then construct the corresponding bifurcation diagram on sc-plane.
Ans. There are two critical points if $s < \frac{1}{16}$ and one critical point if $s = \frac{1}{16}$ and no critical point if $s > \frac{1}{16}$. The bifurcation diagram is the parabola $(2c-5)^2 = 25(1-16s)$ in the $sc-$plan.
19. Consider the differential equation $\frac{dx}{dt} = kx - x^3$. (a) If $k \leq 0$, show that the only critical value $c = 0$ of x is stable. (b) If $k > 0$, show that the critical point $c = 0$ is now unstable but that the critical points $c = \pm\sqrt{k}$ are stable. Thus the qualitative nature of the solutions changes at $k = 0$ as the parameter k increases and so $k = 0$ is a bifurcation point for the differential equation with parameter k.
20. Consider the differential equation $\frac{dx}{dt} = x + kx^3$ containing the parameter k. Analyze the dependence of the number and nature of the critical points on the value of k and construct the corresponding bifurcation diagram.
21. The predator-prey system

$$\frac{dx}{dt} = 5x - x^2 - xy, \qquad \frac{dy}{dt} = -2y + xy$$

in which the prey population $x(t)$ is logistic but the predator population $y(t)$ would(in the absence of any prey) decline naturally. Show that the system has three critical points $(0,0)$, $(5,0)$ and $(2,3)$ with saddle points at the origin and on the positive $x-$axis, and with a spiral sink interior to the first quadrant.

Chapter 3

Total (Or Pfaffian) Differential Equations

3.1 Introduction

In this chapter, we proposed to discuss differential equations with one independent variable and more than one dependent variables.

Definition 3.1 (**Pfaffian Differential Equation:**) Let $u_i,\ i = 1, 2, \cdots, n$ be n functions of some or all of n independent variables $x_1, x_2, \cdots, x_n$. Then $\sum\limits_{i=1}^{n} u_i dx_i$ is called a Pfaffian differential form in n variables and $\sum\limits_{i=1}^{n} u_i dx_i = 0$ is called a Pfaffian differential equation in n variables $x_1, x_2, \cdots, x_n$.

Definition 3.2 (**Total(Pfaffian)Differential Equation for Three Variables:**)

An equation of the form $$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \tag{3.1}$$

is called the Pfaffian Differential Equation in three variables x, y, z.

The equation (3.1) can be directly integrated if there exists a function $u(x, y, z)$ whose total differential du is equal to the left hand number of (3.1). In other cases (3.1) may or may not be integrable. We now proceed to find the condition which P, Q, R must satisfy, so that (3.1) may be integrable. This will be called the condition or criteria of integrability of this single differential equation (3.1).

3.2 Necessary and sufficient conditions for integrability of total(or single) differential equation $Pdx + Qdy + Rdz = 0$.

3.2.1 Necessary condition:

Consider the total (or single) differential equation

$$Pdx + Qdy + Rdz = 0 \qquad \text{where } P, Q, R \text{ are functions of } x, y, z. \tag{3.2}$$

$$\text{Let (3.2) have an integral} \qquad u(x, y, z) = c \tag{3.3}$$

Then total differential du must be equal to $Pdx + Qdy + Rdz$, or to it multiplied by a factor. But, we know that

$$du = (\frac{\partial u}{\partial x})dx + (\frac{\partial u}{\partial y})dy + (\frac{\partial u}{\partial z})dz. \tag{3.4}$$

Since (3.3) is an integral of (3.2), P, Q, R must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$. Therefore, $\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \lambda(x, y, z)$ (say).

$$\lambda P = \frac{\partial u}{\partial x}, \qquad \lambda Q = \frac{\partial u}{\partial y}, \qquad \text{and } \lambda R = \frac{\partial u}{\partial z} \tag{3.5}$$

From the first two equations of (3.5), we get

$$\begin{aligned}
&\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) = \frac{\partial}{\partial x}(\lambda Q)\\
&\text{or, } \lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}\\
&\text{or, } \lambda\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \qquad (3.6)\\
\text{Similarly} \quad &\lambda\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \qquad (3.7)\\
\text{and} \quad &\lambda\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \qquad (3.8)
\end{aligned}$$

Multiplying (3.6)-(3.8) by R, P and Q respectively and adding, we get

$$P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = 0. \tag{3.9}$$

This is therefore, the necessary condition for the integrability of the equation (3.2).

3.2.2 Sufficient condition:

Suppose that the coefficients P, Q, R of (3.2) satisfy the relation (3.9). It will now be proved that this relation gives the required sufficient condition for the existence of an integral of (3.2). For this we show that an integral of (3.2) can be found when relation (3.9) holds. We first prove that if we take $P_1 = \mu P$, $Q_1 = \mu Q$, $R_1 = \mu R$ where μ is any function of x, y and z, the same condition is satisfied by P_1, Q_1, R_1 as by P, Q, R. We have

$$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} - \Big(\mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}\Big)$$

$$\text{or, } \frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y} \tag{3.10}$$

$$\text{Similarly, } \frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z} = \mu\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z} \tag{3.11}$$

$$\text{and } \frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x} = \mu\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) + P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} \tag{3.12}$$

Multiplying (3.10), (3.11) and (3.12) by P_1, Q_1, R_1 respectively, adding and replacing P_1, Q_1, R_1 by μP, μQ, μR respectively in resulting R.H.S., we obtain

$$P_1\Big(\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y}\Big) + Q_1\Big(\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z}\Big) + R_1\Big(\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x}\Big) \tag{3.13}$$

$$= \mu\Big\{P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\Big\} = 0 \text{ using (3.9)} \tag{3.14}$$

Now $Pdx + Qdy$ may be regarded as an exact differential. For if it is not so, then multiplying the equation (3.2) by the integrating factor $\mu(x, y, z)$, we can make it so. Thus there is no loss of generality in regarding $Pdx + Qdy$ as an exact differential. For this the condition is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{3.15}$$

$$\text{Let } V = \int (Pdx + Qdy) \tag{3.16}$$

$$\text{then it follows that } P = \frac{\partial V}{\partial x} \quad \text{ and } Q = \frac{\partial V}{\partial y} \tag{3.17}$$

$$\text{From (3.17), } \frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z \partial x} \quad \textit{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z \partial y}. \tag{3.18}$$

By using the above relation, (3.15), (3.17) and (3.9) gives

$$\frac{\partial V}{\partial x}\Big(\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial R}{\partial y}\Big) + \frac{\partial V}{\partial y}\Big(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\Big) = 0 \qquad \text{or} \qquad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\Big(\frac{\partial V}{\partial z} - R\Big) - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial z} - R\Big) = 0$$

$$\text{or } \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial z} - R\Big) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\Big(\frac{\partial V}{\partial z} - R\Big) \end{vmatrix} = 0.$$

This show that a relation independent of x and y exists between V and $(\frac{\partial V}{\partial z}) - R$. Consequently $(\frac{\partial V}{\partial z}) - R$ can be expressed as a function of z and V alone. That is we can take

$$\frac{\partial V}{\partial z} - R = \phi(z, V). \tag{3.19}$$

$$\text{Now, } Pdx + Qdy + Rdz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + (\frac{\partial V}{\partial z} - \phi)dz, \text{ using (3.16) and (3.19)}$$

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz - \phi dz = dV - \phi dz. \tag{3.20}$$

Thus (3.2) may be written as $dV - \phi dz = 0$ which is an equation in two variables. Hence its integration will gives an integral of the form. Hence the condition (3.9) is sufficient. Thus (3.9) is both the necessary and sufficient condition that (3.2) has an integral.

Theorem 3.1 Prove that the necessary and sufficient condition for integrability of the total differential equation $\mathbf{A}.d\mathbf{r} = Pdx + Qdy + Rdz = 0$ is $\mathbf{A}.\mathbf{curl}\,\mathbf{A} = 0$.

Proof. Given $\mathbf{A}.d\mathbf{r} = Pdx + Qdy + Rdz = 0. \quad (3.21)$

Let $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that $d\mathbf{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} \quad (3.22)$

and $\mathbf{A} = P\hat{i} + Q\hat{j} + R\hat{k} \quad (3.23)$

Then we see that (3.21) is satisfied by usual rule of dot product of two vectors $\mathbf{A}$ and $d\mathbf{r}$. No show that the necessary condition for integrability of (3.21) is

$$P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = 0. \tag{3.24}$$

From vector calculus, we have

$$\mathbf{Curl\,A} = \Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big)\hat{i} + \Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big)\hat{j} + \Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\hat{k} \tag{3.25}$$

Hence, using (3.23) and (3.25) and applying the usual rule of dot product of two vectors, the necessary condition (3.24) may be rewritten as $\mathbf{A}.\mathbf{curl}\,\mathbf{A} = 0$ as desired.

3.3 The conditions for exactness of $Pdx + Qdy + Rdz = 0$.

The given total differential equation is said to be exact if the following three conditions are satisfied.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \tag{3.26}$$

Note that when condition (3.26) are satisfied, the condition for integrability of $Pdx+Qdy+Rdz = 0$, namely,

$$P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = 0 \tag{3.27}$$

is also satisfied for each term of (3.27), vanishes identically.

3.4 Show that the locus of $Pdx + Qdy + Rdz = 0$ is orthogonal to the locus of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{The equation } Pdx + Qdy + Rdz = 0 \tag{3.28}$$

means, geometrically that a straight line whose direction cosines are proportional to dx, dy, dz is perpendicular to a line whose direction cosines are proportional to P, Q, R. As a consequence a point which satisfies (3.28) must move in a direction at right angles to a line whose direction cosines are proportional to P, Q, R. On the other hand, the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3.29}$$

mean geometrically that a straight line whose direction cosines are proportional to dx, dy and dz. Again from the equation (3.29) (i.e. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$), we see that dx, dy and dz are proportional to P, Q and R. Thus (P, Q, R) are the corresponding direction rations of the tangents to the curves at (x, y, z). Thus geometrically the above differential equations represents a system of curves in space such that the direction cosines of the tangent to these curves at any point (x, y, z) are proportional to (P, Q, R).

From the above discussion it follows that the curves traced out by the points that are moving according to the condition (3.28) are orthogonal to the curves traced out by the points that are moving according to the conditions (3.28). The former curves are any of the curves upon the surfaces given by (3.28). Thus geometrically, the curves represented by (3.29) are normal to the surfaces represented by (3.28). In case (3.28) is not integrable, there can not exist a family of surfaces which is orthogonal to all lines that form the locus of (3.29).

3.5 Geometrical Interpretation of $Pdx + Qdy + Rdz = 0$

The given differential equation expresses that the tangent to a curve is perpendicular to a certain line, the direction cosines of this tangent line and another line being proportional to dx, dy, dz and P, Q, R respectively. Suppose that the equation

$$Pdx + Qdy + Rdz = 0 \tag{3.30}$$

satisfies the condition of integrability and that its solution is

$$F(x, y, z, c) = 0. \tag{3.31}$$

Since (3.31) has one arbitrary constant, it represents a single infinity of surfaces. Choosing this constant in an appropriate manner, (3.31) can be made to pass through any given point of space. If a point is moving upon this surface in any direction, its co-ordinates and direction cosines of its path at any moment must satisfy (3.30), since (3.31) is the integral of (3.30). Again for each point (x, y, z) there will be an infinite number of values of dx, dy, dz which will satisfy (3.30). Thus it follows that a point which is moving in such a manner that its co-ordinates and the direction cosines of its path always satisfy (3.30) can pass through any point in an infinity of directions. However, while passing through any point, it must remain on the particular surface given by (3.31) which passes through the point. Thus, infinite number of such possible curves which it can describe through that point must lie on the surface.

3.6 Methods of solution of the Total Differential Equation

In this section, we discuss various type of methods from which a suitable one will be used to obtain the corresponding integral of the total differential equation. At first the conditions of integrability as given by (3.9) should be verified first, then the suitable method for the determination of the corresponding integral as given below will be considered.

3.6.1 Method I(Solution by inspection)

By rearranging the terms of the given equation or by dividing by a suitable functions of x, y, z to reduce some part of the equation into exact differentials and then integrating, the required integral is determined.

Example 3.1 Solve $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$.

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = yz + 2x, \qquad Q = zx - 2z, \qquad R = xy - 2y.\\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\\
&= (yz + 2x)\{(x - 2) - (x - 2)\} + (zx - 2z)(y - y) + (xy - 2y)(z - z) = 0
\end{aligned}$$

Showing that the given total differential equation is integrable. On rearranging, the given equation can be written as

$$\begin{aligned}
&(yzdx + zxdy + xydz) + 2xdx - 2(zdy + ydz) = 0\\
&\Rightarrow d(xyz) + d(x^2) - 2d(yz) = 0\\
&\text{Integrating we get}, \ xyz + x^2 - 2yz = c,
\end{aligned}$$

which is the required general solution, c being an arbitrary constant.

3.6.2 Method II(Solution of homogeneous equation)

The equation $Pdx+Qdy+Rdz = 0$ is called a homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree.

Step I: As usual verify that the given equation is integrable.

Step II: Put $x = zu$, $y = zv$ so that $dx = udz + zdu$ and $dy = zdv + vdz$. Substituting these in the given equation, we solve the equation.

Example 3.2 Solve $2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0$

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get, $P = 2y + 2z$, $Q = -x - z$, $R = 2y - x + z$ and $\sum P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) = 0$. So the condition of integrability is satisfied.
The given equation is homogeneous. Let us consider $x = uz$, $y = vz$ so as to obtain $dx = udz + zdu$, $dy = vdz + zdv$. Then the given equation becomes

$$\begin{aligned}
&z[2(v + 1)du - (u + 1)dv] + (u + 1)(v + 1)dz = 0\\
&\text{or, } 2\frac{du}{u + 1} - \frac{dv}{v + 1} + \frac{dz}{z} = 0
\end{aligned}$$

Integrating we get, $(u+1)^2 z = c(v+1)$ or $(x+z)^2 = c(y+z)$ which is the required general solution and c being an arbitrary constant.

3.6.3 Method III(Use of auxiliary equation)

Let the total differential equation $Pdx + Qdy + Rdz = 0$ be integrable. Then it follows that P, Q, R satisfy

$$P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = 0$$

Now comparing these two equations we have

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{3.32}$$

known as the auxiliary equations. Then the above auxiliary equations is solved using the methods (for solving simultaneous equation of type-II) as described in the chapter-**??**.
Let $u = c_1$, $v = c_2$ (where c_1, c_2 being arbitrary constants) be the integrals obtained by solving equation (3.32). Now we find two functions ϕ and ψ such that the equation may be written by $\phi du + \psi dv = 0$.

In fact as u and v are known, the equation $\phi du + \psi dv = 0$ will reduce to an equation of the form $P_1 dx + Q_1 dy + R_1 dz = 0$ and comparing this with the original equation the values of the function ϕ and ψ are determined and finally with these values of ϕ and ψ, the equation $\phi du + \psi dv = 0$ can be integrated to obtain the required integral of the given equation.

Example 3.3 Solve $xz^3 dx - zdy + 2ydz = 0$.

Solution: Here,

$$xz^3 dx - zdy + 2ydz = 0 \tag{3.33}$$

Comparing (3.33) with $Pdx + Qdy + Rdz = 0$, here

$$P = xz^3,\ Q = -z,\ R = 2y \tag{3.34}$$

Here $P\big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\big) + Q\big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\big) + R\big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\big) = xz^3 \cdot (-1-2) - z \cdot (0 - 3xz^2) + 2y \cdot 0 = -3xz^3 + 3xz^3 = 0$.
Thus the condition of integrability is satisfied.
The auxiliary equations of the given equation are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{3.35}$$

$$\frac{dx}{-1-2} = \frac{dy}{0 - 3xz^2} = \frac{dz}{0} \tag{3.36}$$

$$\frac{dx}{1} = \frac{dy}{xz^2} = \frac{dz}{0} \tag{3.37}$$

Taking the third ratio of (3.37), we have

$$dz = 0 \qquad \text{so that } z = c_1 = u \text{ (say)} \tag{3.38}$$

Taking the first and second ratios of (3.37), we have

$$xz^2 dx - dy = 0 \tag{3.39}$$

Integrating,

$$x^2u^2 - 2y = c_2 = v \text{ (say)} \tag{3.40}$$

$$x^2z^2 - 2y = v \text{ using (3.38)} \tag{3.41}$$

Substituting the value of u and v from (3.37) and (3.38) in $Adu + Bdv = 0$ we get $Adz + B(2xz^2dx + 2x^2zdz - 2dy) = 0$

$$2Bxz^2dx - 2Bdy + (A + 2Bx^2z)dz = 0 \tag{3.42}$$

Comparing (3.33) with (3.42), we get $xz^3 = 2Bxz^2,\ 2y = A + 2Bx^2z$ i.e., $B = \frac{u}{2},\ A = -v$. Substituting these values of A and B in $Adu + Bdv = 0$, we get $-vdu + \frac{udv}{2} = 0 \Rightarrow \frac{dv}{v} = \frac{2du}{u}$. Integrating we get, $\log v = 2\log u + \log c \Rightarrow v = cu^2$ Using (3.38) and (3.41), we get $x^2z^2 - 2y = cz^2$ which is required general solution.

3.6.4 Method IV(General method of solving $Pdx + Qdy + Rdz = 0$ by taking one variable as constant)

Step 1. First verify the integrability condition
Step 2. We now treat one of the variables, say z as a constant i.e., $dz = 0$, then the resulting equation is reduced to

$$Pdx + Qdy = 0 \text{ using} \tag{3.43}$$

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem. The present discussion is for the choice z = constant. For other cases the necessary changes have to be made in the entire procedure.
Step 3. Let the solution of (3.43) by $u(x, y, z) = f(z)$, where $f(z)$ is an arbitrary function of z ($\because z$ = constant). Thus the solution of (3.43) is $u(x, y) = f(z)$.
Step 4. We now differentiate $u(x, y) = f(z)$ $w.r.t.$ x, y, z and then compare the result with the given equation $Pdx + Qdy + Rdz = 0$. After comparing we shall get an equation in two variables f and z. If the coefficient of f or z involve functions of x, y, it will always be possible to remove them with the help of $u(x, y) = f(z)$.
Step 5. Solve the equation got in step 4 and obtain f. Putting this value of f in $u(x, y) = f(z)$, we shall get the required solution of the required equation.

Example 3.4 Solve $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$P = 3x^2, \qquad Q = 3y^2, \qquad R = -(x^3 + y^3 + e^{2z}).$$
$$\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)$$
$$= 0$$

Showing that the condition of integrability is satisfied. Let z be treated as constant, so that $dz = 0$. Then the given equation becomes $3x^2dx + 3y^2dy = 0$. Integrating we get

$$x^3 + y^3 = f(z) \text{ (say) } (\because\ z \text{ is constant}) \tag{3.44}$$

Differentiating (3.44), we get

$$3x^2dx + 3y^2dy - f'(z)dz = 0 \tag{3.45}$$

Comparing (3.45) with the given equation $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$ we have, $f'(z) = x^3 + y^3 + e^{2z} \Rightarrow f'(z) = f(z) + e^{2z}$. Hence its general solution is $f(z) = e^{2z} + ce^z \Rightarrow x^3 + y^3 = e^{2z} + ce^z$, where c is the required solution.

3.6.5 Method V(Exact and homogeneous of degree $n \neq -1$)

Theorem 3.2 If the total differential equation $Pdx + Qdy + Rdz = 0$ is exact and homogeneous of degree $n \neq -1$, then its general solution is given by $Px + Qy + Rz = c$ where c is an arbitrary constant.

Solution: The given total differential equation is $Pdx + Qdy + Rdz = 0$ and so we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \tag{3.46}$$

Also $Pdx + Qdy + Rdz = 0$ is homogeneous of degree $n \neq -1$, so by Euler's theorem on homogeneous functions of x, y, z of degree $n \neq -1$, we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = nP \tag{3.47}$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} = nQ \tag{3.48}$$

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = nR \tag{3.49}$$

Now adding $Pdx + Qdy + Rdz = 0$ with the equations (3.47) - (3.49), we get

$$\begin{aligned} & x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} \\ + \quad & (Pdx + Qdy + Rdz) = (n+1)(Pdx + Qdy + Rdz) = 0 \end{aligned} \tag{3.50}$$

Using the equation (3.46), from (3.50) we get

$$\begin{aligned} & \left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy \\ & \left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0 \end{aligned} \tag{3.51}$$

$$d(Px + Qy + Rz) = 0 \tag{3.52}$$

Integrating we get, $Px + Qy + Rz = c$ is the solution of the total differential equation $Pdx + Qdy + Rdz = 0$ which is exact and homogeneous in x, y, z of degree $n \neq -1$. Hence the theorem.

Example 3.5 Show that $(y + z)dx + (z + x)dy + (x + y)dz = 0$ is exact and homogeneous. Hence solve it.

Solution: Comparing the given total differential equation with $Pdx + Qdy + Rdz = 0$, we get $P = y + z, \ Q = z + x, \ R = x + y$. Then

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1 - 1 = 0 \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 1 - 1 = 0 \Rightarrow \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 1 - 1 = 0 \Rightarrow \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Hence the given total differential equation is exact. Also $P = y(1+\frac{z}{y})$, $Q = z(1+\frac{x}{z})$, $R = x(1+\frac{y}{x})$. So P, Q, R are homogeneous functions of degree 1. Hence the required solution is given by $Px + Qy + Rz = c$ where c being an arbitrary constant. Hence $2xy + 2yz + 2zx = c$ is the required solution.

3.7 Non-integrable Single Differential Equation

Let us consider the single differential equation

$$Pdx + Qdy + Rdz = 0 \tag{3.53}$$

where P, Q, R are functions of x, y, z. If the equation (3.53) does not satisfy the condition of integrability, then there exists no singular relation between x, y, z to satisfy it. However the equation (3.53) represents a family of curves orthogonal to the family represented by $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Thus there exists an infinite number of curves that lie on any given surface and satisfy (3.53). Consequently the method of finding the above mentioned infinite number of curves is equally applicable to integrable equation (3.53).

Let the curves represented by the solution of the non-integrable equation (3.53) lie on the surface represented by

$$u(x, y, z) = c \tag{3.54}$$

where c being an arbitrary constant. Differentiating (3.54) we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{3.55}$$

From (3.55) by eliminating the variables z and dz by using (3.53) and (3.54) we get an equation of the form $Mdx + Ndy = 0$, where M and N are functions of x, y. Solving this we get relation between x, y involving an arbitrary constant and this relation with equation (3.55) give the desire curves i.e., the solution of (3.53).

Example 3.6 Show that there is no single integral of $dz = 2ydx + xdy$. Prove that the curve of the equation that lie in the plane $z = x + y$ lie also on surface of the family $(x-1)^2(2y-1) = c$.

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
& P = 2y, \qquad Q = x, \qquad R = -1. \\
& \therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\
& = 2y \cdot 0 + x \cdot 0 - 1(2-1). \\
& = -1 \neq 0
\end{aligned}$$

Showing that the condition of integrability is not satisfied.
Also from $z = x + y$, we get, $dz = dx + dy$. Then using $2ydx + xdy - dz = 0$ and $dz = dx + dy$, we have,

$$\begin{aligned}
& (2y-1)dx + (x-1)dy = 0 \\
& \frac{2}{x-1}dx + \frac{2}{2y-1}dy = 0
\end{aligned}$$

Integrating we get, $2\log(x-1) + \log(2y-1) = \log c$ i.e., $(x-1)^2(2y-1) = c$.

3.8 Worked Out Example

Example 3.7 Show that $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$ is integrable. Hence solve it.

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = yz + xyz\,, \qquad Q = zx + xyz\,, \qquad R = xy + xyz. \\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\
&= yz(1 + x)\{(x + xy) - (x + xz)\} + zx(1 + y)\{(y + yz) - (y + xy)\} + xy(1 + z)\{(z + xz) - (z + yz)\} \\
&= yz(1 + x)x(y - z) + zx(1 + y)y(z - x) + xy(1 + z)z(x - y) \\
&= xyz\{(1 + x)(y - z) + (1 + y)(z - x) + (1 + z)z(x - y)\} \\
&= xyz[0 - 0] = 0.
\end{aligned}$$

Showing that the condition of integrability is satisfied.
Dividing each term by xyz, the given equation becomes

$$(\frac{1}{x} + 1)dx + (\frac{1}{y} + 1)dy + (\frac{1}{z} + 1)dz = 0$$

Integrating, we get, $\log x + x + \log y + y + \log z + z = c$ or $\log(xyz) + x + y + z = c$, which is the required general solution, where c being an arbitrary constant.

Example 3.8 Show that $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$ is an integrable. Hence solve it.

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = z(z - y)\,, \qquad Q = z(z + x)\,, \qquad R = x(x + y). \\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\
&= z(z - y)(2z + x - x) + z(z + x)(2x + y - 2z + y) + x(x + y)(-z - z) = 0,
\end{aligned}$$

which show that the condition of integrability is satisfied. So it is an integrable.
The auxiliary equations of the given equation are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dy}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{3.56}$$

$$\Rightarrow \quad \frac{dx}{z} = \frac{dy}{x + y - z} = \frac{dz}{-z}. \tag{3.57}$$

From first and third ratios, we get $dx + dz = 0$. Integrating, $x + z = c_1$(constant) or $u = c_1$ where $u = x + z$. Again from (3.57), we have $\frac{dx+dy}{x+y} = \frac{dz}{-z}$. Integrating, $\log(x + y) + \log z = \log c_2 \Rightarrow (x + y)z = c_2$ or $v = c_2$ where $v = (x + y)z$.
Now we wish to find ϕ and ψ s.t the given equation is identical with

$$\phi du + \psi dv = 0 \tag{3.58}$$

Then using $u = c_1$, $v = c_2$ we have from (3.58),

$$\begin{aligned}
&\phi d(x + y) + \psi d(xz + yz) = 0 \\
\Rightarrow \quad &(\phi + z\psi)dx + z\psi dy + (\phi + x + y)dz = 0
\end{aligned}$$

Comparing this equation with the given equation, we get $\phi + z\psi = kz(z - y)$, $z\psi = kz(z + x)$ and $\phi + x + y = kx(x + y)$, where k is non zero constant. Now solving the above relations, we get $\psi = k(x + z) = ku$ and $\phi = kz(z - y) - z\psi = kz(z - y) - kz(z + x) = -ku$. Using the values of ϕ and ψ in (3.58) becomes $-vdu + udv = 0 \Rightarrow \frac{du}{u} - \frac{dv}{v} = 0$. Then integrating we get, $\log u - \log v = \log c \Rightarrow u = cv$ or $x + z = cz(x + y)$ where c being an arbitrary constant and this is the required solution of the given equation.

Example 3.9 Find the orthogonal trajectories on the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ of its intersection with the family of planes parallel to $z = 0$.

Solution: Given surface is

$$f(x, y, z) = x^2 + y^2 - z^2 \tan^2 \alpha = 0 \tag{3.59}$$

and the family of planes parallel to $z = 0$ is

$$z = k \tag{3.60}$$

where k is parameter. Then the system of differential equations of the given curves of intersection of (3.59) and (3.60) is given by

$$2xdx + 2ydy - 2z \tan^2 \alpha dz = 0, \qquad dz = 0 \tag{3.61}$$

Solving these equation for dx, dy, dz, we get

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}.$$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is

$$xdx + ydy - z \tan^2 \alpha dz = 0, \qquad ydx - xdy + 0 \cdot dz = 0 \tag{3.62}$$

Solving these equation for dx, dy, dz, we get

$$\frac{dx}{xz \tan^2 \alpha} = \frac{dy}{yz \tan^2 \alpha} = \frac{dz}{x^2 + y^2}. \tag{3.63}$$

Taking $x, y, 0$ as multipliers, each fraction of (3.63) $= \frac{xdx+ydy}{(x^2+y^2)z \tan^2 \alpha}$. Combining this fraction with last fraction in (3.63), we get

$$\frac{xdx + ydy}{(x^2 + y^2)z \tan^2 \alpha} = \frac{dz}{x^2 + y^2}$$

so that

$$2xdx + 2ydy - 2z \tan^2 \alpha dz = 0.$$

Integrating,

$$x^2 + y^2 - z^2 \tan^2 \alpha = k$$

where k being an arbitrary constant. Choosing $k = 0$, we obtain the given surface (3.59). Taking the first and second fractions of (3.62), we get $\frac{dx}{x} - \frac{dy}{y} = 0$. Integrating, we get, $\log x - \log y = \log c$ or $x = cy$, c being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by $x^2 + y^2 = z^2 \tan^2 \alpha$ and $x = cy$.

Example 3.10 Find the orthogonal projection of the curves on the $xy-$plane which lie on the paraboloid $3z = x^2 + y^2$ and satisfy the differential equation $2dz = (x + y)dx + ydy$.

Solution: Given paraboloid is

$$f(x, y, z) = x^2 + y^2 - 3z = 0 \tag{3.64}$$

The given differential equation is

$$(x + z)dx + ydy - 2dz = 0 \tag{3.65}$$

Comparing (3.65) with $Pdx + Qdy + Rdz = 0$, we have $P = x + y$, $Q = y$ and $R = -2$. Now

$$\begin{aligned} & P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\ &= (x + y)(0 - 0) + y(0 - 1) - 2(0 - 0) = -y \neq 0. \end{aligned}$$

Thus (3.65) does not satisfy the integrability condition. Now differentiability (3.64), we have

$$2xdx + 2ydy - 3dz = 0 \tag{3.66}$$

Then from (3.65) and (3.66), we get

$$\begin{aligned} & 3(x + z)dx + 3ydy - 2(2xdx + 2ydy) = 0 \\ \Rightarrow\ & xdx + ydy - 3zdx = 0 \\ \Rightarrow\ & xdx + ydy = (x^2 + y^2)dx, \text{ using (3.64)} \\ \Rightarrow\ & \frac{2xdx + 2ydy}{x^2 + y^2} = 2dx \end{aligned}$$

Integrating we get, $\log(x^2 + y^2) = \log c + 2x \Rightarrow x^2 + y^2 = ce^{2x} \Rightarrow 3z = ce^{2x}$ (using (3.64)) where c being an arbitrary constant.

3.9 Review Exercises

1 Show that $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ is integrable.

2 Solve the following equations using Method I

(a) $yz\log z dx - zx\log z dy + xydz = 0$, **Ans.** $x\log z = cy$

(b) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$, **Ans.** $x^2z + y^3 = cx$

(c) $\frac{y+z-2x}{(y-x)(z-x)}dx + \frac{z+x-2y}{(z-y)(x-y)}dy + \frac{x+y-2z}{(x-z)(y-z)}dz = 0$, **Ans.** $(x - y)(y - z)(z - x) = c$

(d) $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$, **Ans.** $xy + yz + zx = c(x + y + z)$

(e) $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$, **Ans.** $x^2 + y^2 + z(x + y) = cxy$

3 Solve the following equations using Method II

(a) $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$, **Ans.** $x(y + z) = c(x + y)$

(b) $(x - y)dx - xdy + zdz = 0$, **Ans.** $x^2 - 2xy + z^2 = c$

(c) $yz^2(x^2 - yz)dx + x^2z(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$, **Ans.** $x^2z + y^2x + z^2y = cxyz$

(d) $(2z^2 - xy + y^2)zdx + (2z^2 + x^2 - xy)zdy - (x + y)(xy + z^2)dz = 0$, **Ans.** $(x + y)^2z = c(z^2 - xy)$

4 Solve the following equations using Method *III*
(a) $xz^3dx - zdy + 2ydz = 0$, **Ans.** $x^2z^2 - 2y = cz^2$
(b) $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$, **Ans.** $x^3 + y^3 = e^{2z} + ce^z$
(c) $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$, **Ans.** $xk = x^2 + y^2 + z^2$
(d) $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ **IAS : 1999**, **Ans.** $(x + z)y = k(y + z)$

5 Solve the following equations using Method *IV*
(a) $(y^2 + z^2 + x^2)dx - 2xydy - 2xzdz = 0$, **VU(CBCS):2018** **Ans.** $y^2 + z^2 = x^2 + xk$
(b) $yz \log xdx - zx \log zdy + xydz = 0$, **Ans.** $x \log z = cy$
(c) $zydx + (x^2y - xz)dy + (x^2y - xy)dy + (x^2z - xy)dz = 0$, **CU(H):2016** **Ans.** $x^2(y^2 + z^2 - 2c) = 2xyz$
(d) $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0$, **Ans.** $nx - lz = c(mz - ny)$

6 Solve $2xzdx + zdy - dz = 0$. **Ans.** $x^2 + y - \log z = k$ is the required solution where k is arbitrary constant.

7 Solve $2xzdx + zdy - dz = 0$. **Ans.** $x^2 + y - \log z = c$.

8 Solve $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$. **Ans.** $x^2 - xy + y^2 = cz$.

9 Solve $(x^2 + xy + yz)dx - x(x + z)dy + x^2dz = 0$. **Ans.** $x + z = ce^{\frac{y}{x}}$.

10 Find $f(y)$ such that the total differential equation $\frac{(yz+z)}{x}dx - zdy + f(y)dz = 0$ is integrable. Hence solve it. **VU(CBCS): 2018** **Ans.** $f(y) = k(y + 1)$ and required solution is $xz^k = c(y + 1)$ where k, c are arbitrary constants.

11 Find the general solution of the equation $ydx + (z - y)dy + xdz = 0$ which is consistent with the relation $2x - y - z = 1$. **Ans.** $x^2 + xy - y^2 - y = k$ is the required solution where k is arbitrary constant.

12 Find the system of curves lying on the system of surfaces $xz = c$ and satisfying the differential equation $yzdx + z^2dy + y(z + x)dz = 0$. **Ans.** System of curves are $x = c_1y$ and $xz = c_2$ where c_1, c_2 are arbitrary constant.

13 Show that the curves satisfying the differential equation $xdx + ydy + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}dz = 0$ that lie on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie also on the family of concentric spheres $x^2 + y^2 + z^2 = a^2$.

14 Find the orthogonal trajectories on the conicoid $(x + y)z = 1$ of the conics in which it which it is cut by the system of planes $x - y + z = k$, where k is parameter. **Ans.** $(x + y)z = c$ and $x + c' = z - \frac{1}{6z^3} + \frac{1}{2z}$.

15 Find $f(y)$ if $f(y)dx - zxdy - xy \log ydz = 0$ is integrable. **Ans.** $f(y) = cy$

16 Solve: $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$, **Ans.** $x + z = cz(x + y)$ where c is an arbitrary constant.

17 Solve : $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$. **Ans.** $x^3 + y^3 = e^{2z} + ce^z$ where c is an arbitrary constant.

18 Solve : $(\cos x + e^xy)dx + (e^x + e^yz)dy + e^ydz = 0$. **Ans.** $yce^x + ze^y + \sin x = c$ where c is an arbitrary constant.

19 Solve : $(z + z^2)\cos xdx - (z + z^2)dy + (1 - z^2)(y - \sin x)dz = 0$. **Ans.** $y = \sin x - cze^{-z}$ where c is an arbitrary constant.

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